COMPLEX ANGULAR MOMENTUM DIAGONALIZATION OF THE BETHE-SALPETER STRUCTURE IN GENERAL QUANTUM FIELD THEORY

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Abstract

The Complex Angular Momentum (CAM) representation of (scalar) four-point functions has been previously established starting from the general principles of local relativistic Quantum Field Theory (QFT). Here, we carry out the diagonalization of the general t-channel Bethe–Salpeter (BS) structure of four point functions in the corresponding CAM variable λ_t, for all negative values of the squared-energy variable t. This diagonalization is closely related to the existence of BS-equations for the absorptive parts in the crossed channels, interpreted as convolution equations with spectral properties. The production of Regge poles equipped with factorized residues involving Euclidean three-point functions appears as conceptually built-in in the analytic axiomatic framework of QFT. The existence of leading Reggeon terms governing the asymptotic behaviour of the four-point function at fixed t is strictly conditioned by the asymptotic behaviour of a global Bethe-Salpeter kernel of the theory.

1 Introduction

In a previous paper [1], we have proved that the existence of analytic structures in the Complex Angular Momentum (CAM) variables is a general property of Quantum Field Theories (QFT) satisfying the basic principles [2] of local commutativity, Poincaré invariance, spectral condition and temperate ultraviolet behaviour. Considering four-point functions $H([k])$ of general relativistic scalar fields in a given two-field channel called the "$t-$channel", we have in fact shown that appropriate Laplace-type transforms $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ of H are *holomorphic functions of a CAM variable* λ_t, dual to the (off-shell) scattering angle Θ_t of the $t-$channel. The analyticity domain which was obtained for $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ is a half-plane of the form $\Re e\lambda_t > N_H$, where the number N_H ($N_H > 0$) corresponds to a certain "degree of temperateness" of $H([k])$ at large momenta. This domain was obtained for all negative values of the squared total energy t of the channel considered.

It is the purpose of our program to investigate under which general conditions the transforms $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ of H may admit meromorpic continuations in a domain of the joint complex variables (t, λ_t) which constitutes a "bridge" between a) the "primitive set " $\{(t, \lambda_t);\ t < 0;\ \text{Re}\ \lambda_t > N_H\}$ obtained in [1] (and from which poles are excluded), and b) a complex neighbourhood of a real set of the form $\{(t, \lambda_t);\ 0 < t < t_0;\ \lambda_t > N(t)\}$ in which possible poles of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ in the variables (t, λ_t) (namely "Regge poles") might interpolate bound states of H, supposed to be present in that real set (at integral values $\lambda_t = 0, 1, \ldots$).

In the framework of analyticity properties implied by the general principles of QFT, the existence of a meromorphic continuation of H exhibiting real poles at $t > 0$ under the two-particle threshold (resp. complex poles in the two-particle second sheet of the $t-$plane) is a consequence of the general Bethe-Salpeter structure of QFT, in which the additional postulate of "Asymptotic Completeness" (or "off-shell unitarity") plays an essential role [8,9].

As we have announced it in [3], we shall show that the general Bethe-Salpeter (BS) structure of QFT is also operational for displaying the *Regge pole interpolation of bound states* as a consequence of the basic principles of QFT. In order to perform this program, it is necessary

i) to extend the analysis of the general BS structure of field theory so as to include the complex angular momentum variables, which requires in a first step that we restrict ourselves to the range $t < 0$, along the line of [1];

ii) to justify the analytic continuation of this structure at $t > 0$.

Part i) is the object of the present paper, while part ii) will be treated in a further paper.

The contents of this paper can be described as follows. We start from a $t-$channel Bethe–Salpeter type structural equation for the four-point function of scalar fields; the latter is considered in the space of Euclidean energy-momenta which is fully contained in the primitive axiomatic analyticity domain. Such an integral equation involves a ("regularized two-particle-irreducible") Bethe-Salpeter-type kernel B whose introduction is completely justified on the basis of the general principles of local QFT and (as proved in [8,9]) the complete two-field structure of the theory in the $t-$channel is encoded in this equation. These results and the corresponding structural equation for the absorptive parts in the crossed s and $u-$channels [8,10] are recalled in Sec 2. In Sec.3, we work out various aspects of these structures in the mass variables and angular variables and ultimately in the CAM variable λ_t. After having written the BS-equation in the mass variables and angular variables we perform its partial-wave decomposition in the Euclidean region in terms of "off-shell Euclidean partial-waves". Next we are interested in performing the analytic continuation of the BS-equation from Euclidean subspace to the real Minkowskian subspace *by travelling in the complex angular variables of the $t-$channel inside the "enlarged analyticity domain" obtained in* [1] until reaching the spectral sets of the crossed s and $u-$channels. This allows one to give a new approach to the ($t-$channel) BS-equations for the symmetric and antisymmetric combinations of the s and

$u-$channel absorptive parts. In this study, an important role is played by an appropriate class of holomorphic kernels on the complexified hyperboloid, called "perikernels", and by the "$\star^{(c)}-$convolution product" of such kernels [11], which is diagonalized by the Laplace-type transformation L_d introduced in [1]. It follows that the BS-equations for the s and $u-$channel absorptive parts admit "t-channel Laplace-type transforms" which are themseves Fredholm-type integral equations depending analytically of the CAM variable λ_t in a half-plane of the form $\Re e\lambda_t > N$. In view of the results of [1] ("Froissart-Gribov equalities"), these integral relations provide Carlsonian interpolations *in this half-plane* of the corresponding BS-equations for the even and odd Euclidean partial waves. In the subsequent part of this paper (Sec 4), the theory of Fredholm resolvent equations and specifically the $\frac{\mathcal{N}}{\mathcal{D}}$ method, $\mathcal{N}$ and $\mathcal{D}$ being regarded as analytic functions of both variables t and λ_t, are shown to provide a general framework for generating Regge poles equipped with factorized residues involving Euclidean three-point functions. The existence of corresponding leading "Reggeon terms" in the asymptotic behaviour of H at large s and fixed values of t and of the mass variables necessitates the meromorphic continuation of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ in a strip of the form $N_B(t) < \mathrm{Re}\ \lambda_t < N_H$: this property is shown to be strictly conditioned by the existence of a "two-particle irreducible kernel" B whose behaviour in the complex s (or u) plane at fixed values of t and of the mass variables is bounded by $s^{N_B(t)}$, with $N_B(t)$ smaller than N_H. An Appendix is devoted to a survey of the Fredholm theory in complex space with complex parameters [8,9,13,17], which covers the various versions of BS-equations encountered in Sec.3 and 4; it establishes the corresponding analyticity properties and bounds of their solutions in the relevant variables.

2 Bethe–Salpeter-type structure of four-point functions in the t-channel: general properties

We are concerned with relativistic local field theory in $(d+1)-$ dimensional space-time, with $d \geq 1$ throughout this section; in the rest of the paper, the interesting structures are relevant for all $d \geq 2$. As our main object of study, we consider a general four-point function in complex momentum space $\mathbb{C}^{d+1}$, denoted by $H\left([k]\right)$, which describes the interaction of two local (and mutually local) fields ϕ_1 and ϕ_2; we specify a certain "t-channel", with squared-total energy $t = \left(k_1 + k_2\right)^2$ as associated with "two-field states" of the form $\left(\int \tilde{\phi}_1\left(k_1\right) \tilde{\phi}_2\left(k_2\right) f\left(k_1, k_2\right) dk_1\ dk_2\right) |\Omega\rangle$, $|\Omega\rangle$ being the vacuum state of this field theory and f any appropriate test-function in momentum space. Following [1], we adapt our notations to this $t-$channel by putting $[k] = \left(k_1, k_2;\ k_1', k_2'\right)$, with $k_1 + k_2 = k_1' + k_2' = K$; k_i and k_i' describe incoming and outgoing energy-momentum vectors carried by the field $\phi_i\,(i = 1, 2)$. We moreover assume that the function $H\left([k]\right)$ is *amputated* from the four external propagator factors $\Pi_i\left(k_i\right), \Pi_i\left(k_i'\right), i = 1, 2,$ Π_i denoting the (complete) two-point function of the field ϕ_i. We introduce the incoming and outgoing relative energy-momentum vectors $Z = \frac{k_1 - k_2}{2}$, $Z' = \frac{k'_1 - k'_2}{2}$ and we also write $H([k]) \equiv H(K; Z, Z')$ as a function of the three independent complex vectors K, Z, Z'.

We refer to [1] for a short survey of the axiomatic analyticity properties of $H([k])$. A basic fact is that, for any fixed Lorentz frame LF with energy-momentum coordinates $K = \left(K^{(0)}, \overrightarrow{K}\right)$, $Z = \left(Z^{(0)}, \overrightarrow{Z}\right)$, $Z' = \left(Z'^{(0)}, \overrightarrow{Z'}\right)$, the function $H([k])$ is holomorphic for all *real or complex* values of the energy-components $K^{(0)}, Z^{(0)}, Z'^{(0)}$ and *real* values of the momentum-components $\overrightarrow{K}, \overrightarrow{Z}, \overrightarrow{Z'}$, *except on "cuts"* which are defined by *"spectral sets"* in the various mass and channel-energy variables

$$\zeta_1 = k_1^2 = \left(\frac{K}{2} + Z\right)^2, \ \zeta_2 = k_2^2 = \left(\frac{K}{2} - Z\right)^2,$$

$$\zeta'_1 = {k'}_1^2 = \left(\frac{K}{2} + Z'\right)^2, \ \zeta'_2 = {k'}_2^2 = \left(\frac{K}{2} - Z'\right)^2,$$

$$s = (k_1 - k'_1)^2 = (Z - Z')^2, \ u = (k_1 - k'_2)^2 = (Z + Z')^2, \ t = K^2.$$

In each of these expressions, the notation k^2 means ${k^{(0)}}^2 - \overrightarrow{k}^2$ for the corresponding vector $k = \left(k^{(0)}, \overrightarrow{k}\right)$ varying in $\mathbb{C}^{d+1}$. The so-called *"absorptive parts"* of H are the discontinuities of $H([k])$, denoted by $\Delta_s H$, $\Delta_u H$, $\Delta_t H$, across the cuts defined respectively by the spectral sets $\Sigma_s (s \geqslant s_0)$, $\Sigma_u (u \geqslant u_0)$, $\Sigma_t (t \geqslant t_0)$; s_0, t_0, u_0 denote the corresponding mass thresholds of these spectral sets (here supposed to be strictly positive).

It is worthwhile to note that for all *massive* field theories *this subset D_{LF} of the primitive axiomatic analyticity domain contains the whole corresponding "Euclidean subspace"* of complex energy-momentum space, namely the set of all complex configurations (K, Z, Z') whose energy-components $(K^{(0)}, Z^{(0)}, Z'^{(0)})$ are purely imaginary and whose momentum components $\left(\overrightarrow{K}, \overrightarrow{Z}, \overrightarrow{Z'}\right)$ are real in the Lorentz frame LF considered.

Another aspect which plays an essential role in the following concerns *"temperateness properties"* of the four-point function $H([k])$, which we assume to be of the form specified in formula (3.1) of [1]. These properties imply bounds of the following form

$$|H(K; Z, Z')| \leqslant C_{loc}(K, Z, Z')(1 + \|K\|)^{N_H}(1 + \|Z\|)^{N_H}(1 + \|Z'\|)^{N_H}, \quad (2.1)$$

where N_H is a fixed power ($N_H \geqslant 0$) and the notation $\|k\|$ stands for the norm of the complex vector $k = (k^{(0)}, \cdots, k^{(d)})$, namely $\|k\|^2 = \sum_{0 \leq i \leq d} |k^{(i)}|^2$; the function C_{loc} includes an inverse power dependence with respect to the distance of the point (K, Z, Z') from the union of the spectral sets, which takes into account the distribution character of the boundary values of $H([k])$ on the reals. The bound (2.1) holds uniformly in the Euclidean subspace and moreover in the following parts of the axiomatic analyticity domain, which are of basic use for the BS-structure:

a) the subset D_{LF} used in [8,9],

b) The union of the complex domains $\underline{D}_{(w,w',\rho')}$ in one vector $\underline{k}$ (for all w, w', ρ') introduced in [1] (see Propositions 3 and 4 of the latter).

2.1 A recall of axiomatic results on the Bethe-Salpeter structure of four-point functions

In [8,9], it has been shown that for massive QFT's satisfying the postulate of *"asymptotic completeness of two-particle states"* (or "off-shell unitarity in the two particle spectral region") and an additional regularity assumption in the energy-variable t (for $t \geq t_0$), the two-particle t-channel analytic structure of $H([k])$ is entirely encoded in any Bethe–Salpeter-type integral equation of the following form

$$H(K;Z,Z') = B(K;Z,Z') + \int_{\Gamma(K)} B(K;Z,Z'')\, H(K;Z'',Z')\, G(K;Z'')\; dZ'' \tag{2.2}$$

where the "Bethe–Salpeter kernel" $B([k]) \equiv B(K;Z,Z')$ is a four-point function satisfying the same axiomatic analyticity properties and bounds of the form (2.1) [1] as $H([k])$ and in addition the property of *"two-particle irreducibility in the t-channel"*. This means that the corresponding absorptive part (or discontinuity) $\Delta_t B$ of B vanishes in the two-particle region, which is of the form $\left\{t; t_0 = (m_1+m_2)^2 \leqslant t < M^2\right\}, m_1$ and m_2 being the masses of the "lowest poles" $\left(k_i^2 = m_i^2,\ i=1,2\right)$ in the respective propagators $\Pi_1(k_1), \Pi_2(k_2)$ of ϕ_1 and ϕ_2.

As a matter of fact, these poles are present in the function $G(K,Z'')$ under the integral of Eq.(2.2), which represents a "regularized double-propagator" with respect to the internal $(d+1)-$momenta $k_i'' = \dfrac{K}{2} \pm Z'', i = 1,2$, of the following form:

$$G(K,Z'') = i\Pi_1^{(\mathrm{reg})}\left(\frac{K}{2}+Z''\right).\Pi_2^{(\mathrm{reg})}\left(\frac{K}{2}-Z''\right); \tag{2.3}$$

$\Pi_i^{(\mathrm{reg})}$ $(i=1,2)$ is a regularized form of Π_i obtained by multiplying the latter by a suitable Pauli–Villars-type factor, equal to 1 on the mass shell. The only required property is that $\Pi_i^{(\mathrm{reg})}$ be a Lorentz-invariant function, namely a function of the squared-mass complex variable $\zeta_i = k_i^2$ having the same pole $\zeta_i = m_i^2$ (with the same residue) as Π_i, analytic in the same cut-plane of the

[1] These analyticity properties and bounds of B are implied by those of H, except for possible "CDD poles" in K corresponding to the Fredholm alternative, which can be excluded for t negative and $|t|$ sufficiently large under a suitable choice of the regularized propagators $\Pi_i^{(reg)}$ (see Appendix, proposition A1).

form $\mathbb{C}\backslash\{m_i^2\}\backslash[M_i^2,+\infty[$ and satisfying uniform bounds of the following form

$$\left|\Pi_i^{(\mathrm{reg})}(k_i)\right| \leqslant \left[c(1+\|k_i\|^2)\right]^{-r}, \tag{2.4}$$

with r sufficiently large and $c<1$, for k_i Euclidean (i.e. $\zeta_i = -\|k_i\|^2$).

The integration cycle $\Gamma(K)$ in Eq. (2.2) is the Euclidean subspace $E_{d+1} = \left\{Z'' = \left(iY''^{(0)}, \overrightarrow{X}''\right)\right\}$, when K, Z and Z' are themselves taken in E_{d+1}.

If r is sufficiently large with respect to N, Eq. (2.2) is a genuine Fredholm integral equation (depending analytically on the vector parameter K) which allows one to define $B([k])$ in terms of $H([k])$, with B also satisfying bounds of the form (2.1). In [8,9], Eq. (2.2) has been shown to extend by analytic continuation to all configurations (K,Z,Z') in the subset D_{LF} of the axiomatic domain, provided $\Gamma(K)$ is distorted from its initial situation $E_{d+1} = E_{d+1}(LF)$ in order to remain in the analyticity domain of the integrand and to have an infinite part parallel to E_{d+1}. (Of course, by using the Lorentz invariance of the axiomatic domain, Eq.(2.2) is also shown to sweep similarly the subsets D_{LF} associated with all Lorentz frames LF). In the following, we shall call *"Feynman convolution"* and denote by $(B \circ_t H)(K;Z,Z')$ the integral at the r.h.s. of Eq.(2.2), which enjoys the axiomatic analyticity properties of a four-point function. Eq.(2.2) can thus be rewritten in short:

$$H = B + B \circ_t H, \tag{2.2'}$$

The analysis of [8,9] relies on the exploitation of Eq. (2.2) as a Fredholm resolvent integral equation on the space $\Gamma(K)$, depending analytically of the (vector) parameter K. In particular, the result of this analysis is that $H([k])$ can be written as a ratio of the following form

$$H(K;Z,Z') = \frac{\mathcal{N}_B(K;Z,Z')}{\mathcal{D}_B(K^2)} \tag{2.5}$$

where $\mathcal{N}_B(K;Z,Z')$ is a four-point function and $\mathcal{D}_B\left(K^2\right)$ is a two-point function, defined in terms of B and G through the appropriate Fredholm series. $\mathcal{N}_B$ and $\mathcal{D}_B$ are proven to admit analytic continuations in the variable $t=K^2$ across the two-particle region $t_0 \leqslant t < M^2$ in a ramified domain around the two-particle threshold $t=t_0$. The zeros of $\mathcal{D}_B\left(K^2\right)$ correspond to poles of $H([k])$, interpreted as bound states or resonances.

2.2 Bethe-Salpeter equations for the absorptive parts in the crossed channels

In the present paper, we shall not exploit Eq. (2.2) for the latter meromorphic structure in t occurring at $\mathrm{Re}\, t > 0$, but for its remarkable implications in the crossed channels, namely the s and u-channels. In fact, it has been shown in [8,10] that Eq. (2.2) implies corresponding integral relations between the absorptive parts $\Delta_s H, \Delta_u H$ of H and $\Delta_s B, \Delta_u B$ of B, which need to be recalled with some care. In these relations, the Feynman convolution $\circ_t$ is replaced by a *composition product* $\diamondsuit_t$ involving integration on a certain *compact* set specified below. We now write these relations in the following concise form, before giving their detailed meaning and interpretation:

$$\Delta_s H = \Delta_s B + \Delta_s B \diamondsuit_t \Delta_s H + \Delta_u B \diamondsuit_t \Delta_u H, \tag{2.6}$$

$$\Delta_u H = \Delta_u B + \Delta_s B \diamondsuit_t \Delta_u H + \Delta_u B \diamondsuit_t \Delta_s H; \tag{2.7}$$

In the latter, the absorptive part $\Delta_s H$ (and similarly for $\Delta_s B$) is specified as follows:

$$\Delta_s H(K;Z,Z') = i \left[\lim_{\substack{\mathrm{Im}\,(Z-Z')^2=-\varepsilon \\ \varepsilon>0 \quad \varepsilon\to 0}} H(K;Z,Z') - \lim_{\substack{\mathrm{Im}\,(Z-Z')^2=\varepsilon \\ \varepsilon>0 \quad \varepsilon\to 0}} H(K;Z,Z')\right] \tag{2.8}$$

The support of $\Delta_s H$ (or $\Delta_s B$), which is the spectral set

$$\Sigma_s = \{(K,Z,Z');\ k_1 - k_1' = Z - Z'\,\text{real},\quad (Z-Z')^2 \geq s_0\}$$

is contained in the union of the following two disjoint sets

$$\Sigma_s^+ = \left\{(K;Z,Z');\ Z-Z'\,\text{real},\ Z-Z' \in V^+\right\},$$

$$\Sigma_s^- = \left\{(K;Z,Z');Z-Z'\,\text{real},\ Z-Z' \in V^-\right\},$$

where V^+ (resp. V^-) denotes the open forward (resp. backward) cone: $k^2 \equiv {k^{(0)}}^2 - \vec{k}^2 > 0,\ k^{(0)} > 0$ (resp. $k^{(0)} < 0$).

We shall call $\Delta_s^+ H$ and $\Delta_s^- H$ (resp. $\Delta_s^+ B$ and $\Delta_s^- B$) the restrictions of $\Delta_s H$ (resp. $\Delta_s B$) to the corresponding disjoint sets Σ_s^+ and Σ_s^-.

The absorptive part $\Delta_u H$ (and similarly for $\Delta_u B$) is specified as follows:

$$\Delta_u H(K;Z,Z') = i \left[\lim_{\substack{\mathrm{Im}\,(Z+Z')^2=-\varepsilon \\ \varepsilon>0 \quad \varepsilon\to 0}} H(K;Z,Z') - \lim_{\substack{\mathrm{Im}\,(Z+Z')^2=\varepsilon \\ \varepsilon>0 \quad \varepsilon\to 0}} H(K;Z,Z')\right] \tag{2.9}$$

The support of $\Delta_u H$ (or $\Delta_u B$), which is the spectral set

$$\Sigma_u = \{(K,Z,Z');\ k_1 - k_2' = Z + Z'\ \text{real},\ (Z+Z')^2 \geq u_0\},$$

is contained in the union of the following two disjoint sets

$$\Sigma_u^+ = \left\{(K;Z,Z');\ Z+Z'\,\text{real},\ Z+Z' \in V^-\right\},$$

$$\Sigma_u^- = \left\{(K;Z,Z');\ Z+Z'\,\text{real},\ Z+Z' \in V^+\right\}.$$

The corresponding restrictions of $\Delta_u H$ (resp.$\Delta_u B$) to Σ_u^+ and Σ_u^- are called $\Delta_u^+ H$ and $\Delta_u^- H$ (resp.$\Delta_u^+ B$ and $\Delta_u^- B$).

For all complex configurations $(K; Z, Z')$ in Σ_s^+ one introduces the "double-cone"

$$\diamondsuit\,(Z, Z') = \left\{Z'';\ Z - Z''\ \text{real},\ Z - Z'' \in V^+,\ Z'' - Z' \in V^+\right\}$$

and defines the composition product $\diamondsuit_t$ as follows:

$$(\Delta_s B\ \diamondsuit_t\ \Delta_s H)\,(K; Z, Z') = \int_{\diamondsuit(Z,Z')} \Delta_s^+ B\,(K; Z, Z'')\,\Delta_s^+ H\,(K; Z'', Z')\,G\,(K; Z'')\,dZ'' \tag{2.10}$$

and similarly

$$(\Delta_u B\ \diamondsuit_t\ \Delta_u H)\,(K; Z, Z') = \int_{\diamondsuit(-Z',-Z)} \Delta_u^- B\,(K; Z, Z'')\,\Delta_u^+ H\,(K; Z'', Z')\,G\,(K; Z'')\,dZ'' \tag{2.11}$$

It is important to note (in view of the distribution character of the absorptive parts $\Delta_{s,u} H$ and $\Delta_{s,u} B$ in the corresponding variables s and u respectively) that these composition products are always well-defined in the sense of distributions in view of the compactness of the integration region.

Eqs. (2.10) and (2.11) specify completely the meaning of formula (2.6) in the case when $Z - Z' \in V^+$.

The complete specification of formulae (2.6) and (2.7) for configurations (K, Z, Z') contained respectively in Σ_s^-, Σ_u^+ and Σ_u^- is obtained by changing the integration region $\diamondsuit\,(Z, Z')$ or $\diamondsuit\,(-Z', -Z)$ of (2.10), (2.11) into an appropriate region of the form $\diamondsuit\,(\varepsilon Z, \varepsilon' Z')$ or $\diamondsuit\,(\varepsilon' Z', \varepsilon Z)$, with $\varepsilon, \varepsilon' = \pm 1$ and by picking-up the corresponding relevant parts $\Delta_{s,u}^{\pm} H$ and $\Delta_{s,u}^{\pm} B$ whose choice is dictated by the consistency of support properties.

The derivation of formulae (2.6), (2.7), which is obtained by taking the absorptive parts Δ_s and Δ_u of both sides of Eq. (2.2), relies on the following discontinuity formulae; for any pair of four-point functions (F_1, F_2), one has:

$$\Delta_s\,(F_1 \circ_t F_2) = \Delta_s F_1\ \diamondsuit_t\ \Delta_s F_2 + \Delta_u F_1\ \diamondsuit_t\ \Delta_u F_2 \tag{2.12}$$

and

$$\Delta_u\,(F_1 \circ_t F_2) = \Delta_s F_1\ \diamondsuit_t\ \Delta_u F_2 + \Delta_u F_1\ \diamondsuit_t\ \Delta_s F_2 \tag{2.13}$$

These formulae have been derived in [8,10] by performing suitable distortions of the integration cycle $\Gamma(K)$ (starting from E_{d+1}) in the Feynman-convolution product $(F_1 \circ_t F_2)(K; Z, Z') = \int_{\Gamma(K)} F_1(K; Z, Z'') F_2(K; Z'', Z') G(K; Z'') dZ''$. These distortions, which are performed in the energy variable ${Z''}^{(0)}$ inside the axiomatic domain of the integrand, lead one to fold the cycle $\Gamma(K)$ around the supports of the absorptive parts $\Delta_{s,u} F_1$ and $\Delta_{s,u} F_2$ in limiting situations $\Gamma(K) = \Gamma_{s\pm}(K)$ (resp. $\Gamma_{u\pm}(K)$) where $s = \Re e s \pm i\eta$ (resp. $u = \Re e u \pm i\eta$), with η positive and tending to zero. Taking the discontinuities Δ_s (resp. Δ_u)

between the convolution integrals on $\Gamma_{s+}(K)$ and $\Gamma_{s-}(K)$ (resp. $\Gamma_{u+}(K)$ and $\Gamma_{u-}(K)$) then reduces the integration cycle to the compact cycle with support $\diamondsuit\,(\varepsilon Z, \varepsilon' Z')$ previously defined.

A nice property enjoyed by the previous formulae (2.12), (2.13) is the following

Additivity property of spectral sets:

If two four-point functions $F_1\,([k])\,, F_2\,([k])$ have their spectral sets Σ_s and Σ_u specified respectively by the conditions $s \geq s_1,\ u \geq u_1$ and $s \geq s_2,\ u \geq u_2$, then the following support properties hold, *inside each connected component* $\Sigma_s^+, \Sigma_s^-, \Sigma_u^+, \Sigma_u^-$ of the spectral sets:

a) The support of $(\Delta_s F_1 \diamondsuit_t \Delta_s F_2)\,(K, Z, Z')$ is contained in the set

$$\left\{(K, Z, Z')\,;\ (Z - Z')^2 \geq (\sqrt{s_1} + \sqrt{s_2})^2\right\}.$$

This directly follows from the definition (2.10) of $\diamondsuit_t$ for $\Delta_s F_1 \diamondsuit_t \Delta_s F_2$ by using the fact that $Z - Z' = (Z - Z'') + (Z'' - Z')$, with $(Z - Z'')^2 \geq s_1$ and $(Z'' - Z')^2 \geq s_2$ ($Z - Z'$ can be either in V^+ or in V^-).

b) Similarly, the supports of $\Delta_u F_1 \diamondsuit_t \Delta_u F_2,\ \Delta_s F_1 \diamondsuit_t \Delta_u F_2,\ \Delta_u F_1 \diamondsuit_t \Delta_s F_2$ are respectively contained in the sets defined by the conditions:

$$\begin{aligned}
(Z - Z')^2 &\geq (\sqrt{u_1} + \sqrt{u_2})^2\,, \\
(Z + Z')^2 &\geq (\sqrt{s_1} + \sqrt{u_2})^2\,, \\
(Z + Z')^2 &\geq (\sqrt{u_1} + \sqrt{s_2})^2
\end{aligned}$$

(with all the possibilities $Z - Z' \in V^{\pm}$ and $Z + Z' \in V^{\pm}$).

Formulae (2.6) and (2.7) appear as a pair of coupled Fredholm-type equations (depending on K) which we call *"Bethe–Salpeter equations for the crossed-channel absorptive-parts"*. A remarkable feature of these equations, which is due to the "additivity property of spectral sets", is the following

Finiteness property of the Bethe-Salpeter equations for crossed-channel absorptive-parts (theorem 1 of [8]):

In any bounded region of the variables s and u, the Bethe-Salpeter equations (2.6) and (2.7) for $\Delta_s H$ and $\Delta_u H$ can be solved explicitly by a finite *number of $\diamondsuit_t$-composition-products.*

In fact, by applying the standard iteration procedure to Eq. (2.6), one gets the following relations, written for simplicity in the case when $s_0 = u_0$:

For $s < 4s_0$, $\qquad \Delta_s H = \Delta_s B.$

For $s < 9s_0$, $\qquad \Delta_s H = \Delta_s B + \Delta_s B \diamondsuit_t \Delta_s B\ +\ \Delta_u B \diamondsuit_t \Delta_u B.$

For $s < 16s_0$,

$$\Delta_s H = \Delta_s B + \Delta_s B \diamondsuit_t \Delta_s B + \Delta_u B \diamondsuit_t \Delta_u B + \Delta_s B \diamondsuit_t \Delta_s B \diamondsuit_t \Delta_s B$$

$$+\Delta_s B \diamondsuit_t \Delta_u B \diamondsuit_t \Delta_u B + \Delta_u B \diamondsuit_t \Delta_s B \diamondsuit_t \Delta_u B + \Delta_u B \diamondsuit_t \Delta_u B \diamondsuit_t \Delta_s B. \tag{2.14}$$

etc...

Eq. (2.7) is solved by similar expressions for $\Delta_u H$.

3 Transferring the Bethe-Salpeter structure from complex momentum space to mass and complex-angular-momentum variables

In this section, we shall derive alternative versions of the Bethe-Salpeter equations (2.2) and (2.6),(2.7) in which the complex energy-momentum vectors are replaced 1) by *squared-mass and complex-angular variables* and 2) by *squared-mass and complex-angular-momentum (CAM) variables*. Throughout these two steps, the four-point function $H([k])$ is considered as given with its axiomatic analyticity properties and temperate bounds of the form (2.1), which imply (according to the results of [1]) two corresponding sets of properties in the complex-angular and CAM variables. At each step, analogous properties of the Bethe-Salpeter kernel B will appear as derived from those of H by applying the Fredholm method or *"$\mathcal{N}/\mathcal{D}$-method"* in complex-space to the corresponding version of the Bethe-Salpeter equation. The common point to these various versions will be the occurrence of an integration-space involving radial and longitudinal variables (ρ, w), equivalent to "squared-mass variables" $\zeta \equiv (\zeta_1, \zeta_2)$, as described below. A unified presentation of the relevant results of the $\mathcal{N}/\mathcal{D}$-method in complex-space, applicable to these various versions is summarized in the Appendix .

We now assume that $d \geq 2$ and fix once for all the total energy-momentum vector K of the t-channel in such a way that K is space-like (i.e. $t < 0$); we choose a system of space-time coordinates such that $K = \left(0, 0, ..., 0, \sqrt{-t}\right)$. For real or complex vectors $k = (k^{(0)}, k^{(1)}, \ldots, k^{(d)})$, $k' = (k'^{(0)}, k'^{(1)}, \ldots, k'^{(d)})$, the Minkowskian scalar product is $k \cdot k' = k^{(0)}k'^{(0)} - k^{(1)}k'^{(1)} - \cdots k^{(d)}k'^{(d)}$ and $k^2 \equiv k \cdot k$.

We adopt the parametrization of the vector variables Z, Z' in terms of *radial, longitudinal and angular variables* ρ, w, z and ρ', w', z' as in [1], Eq. (2.2), namely

$$Z = \rho z + wK \quad , \quad Z' = \rho' z' + w' K, \tag{3.1}$$

where the vectors $z = \left(z^{(0)}, z^{(1)}, ..., z^{(d-1)}, 0\right)$, $z' = \left(z'^{(0)}, z'^{(1)}, ..., z'^{(d-1)}, 0\right)$ are such that:

$$z \cdot K = z' \cdot K = 0 \qquad \text{and} \qquad z^2 = z'^2 = -1. \tag{3.2}$$

The radial and longitudinal variables (ρ, w) (resp. (ρ', w')) can be equivalently replaced by the "*squared-mass variables*" $\zeta = (\zeta_1, \zeta_2)$, resp. $\zeta' = \left(\zeta_1', \zeta_2'\right)$, where

$$\zeta_{1,2} = \left(\frac{K}{2} \pm Z\right)^2 = -\rho^2 + (w \pm \frac{1}{2})^2 t,\ \zeta_{1,2}' = \left(\frac{K}{2} \pm Z'\right)^2 = -{\rho'}^2 + (w' \pm \frac{1}{2})^2 t \tag{3.3}$$

and one has:

$$\rho^2 = \frac{\Lambda(\zeta_1, \zeta_2, t)}{4t}, \qquad \rho'^2 = \frac{\Lambda\left(\zeta_1', \zeta_2', t\right)}{4t}, \tag{3.4}$$

with

$$\Lambda(\alpha, \beta, \gamma) = (\alpha - \beta)^2 - 2(\alpha + \beta)\gamma + \gamma^2 \tag{3.5}$$

and

$$w = \frac{\zeta_1 - \zeta_2}{2t} \quad , \quad w' = \frac{\zeta_1' - \zeta_2'}{2t}. \tag{3.6}$$

The variables ρ, w, ρ', w' will always be real, with $\rho > 0, \rho' > 0$, which means that (ζ, ζ') varies in the real region $\Delta_t \times \Delta_t$, where Δ_t is the parabolic region of (negative) "Euclidean squared-masses" (see fig. 1 of [1])

$$\Delta_t = \{\zeta = (\zeta_1, \zeta_2)\,; \Lambda(\zeta_1, \zeta_2, t) < 0\}.$$

On the contrary, the vector variables z, z' or "*hyperbolic angular variables*" are allowed to vary on the whole *complex* hyperboloid $X_{d-1}^{(c)}$ with equation $z^2 = z'^2 = -1$ (in the subspace $z^{(d)} = {z'}^{(d)} = 0$) and one introduces the variable $\cos\Theta_t = -z.z'$, Θ_t being interpreted as the off-shell (complex) scattering angle in the t-channel.

3.1 Bethe–Salpeter equation in the mass variables and angular variables - Partial-wave decomposition in the Euclidean region

In this subsection, we shall assume that the vectors Z, Z' remain *in the Euclidean subspace* E_{d+1} which in the parametrization (3.1), (3.2) corresponds to choosing the complex vectors z, z' on a unit sphere of dimension $d-1$, called "the Euclidean sphere" S_{d-1} of $X_{d-1}^{(c)}$, namely

$$z = \left(iy^{(0)}, x^{(1)}, ..., x^{(d-1)}, 0\right),\ z' = \left(i{y'}^{(0)}, {x'}^{(1)}, ..., {x'}^{(d-1)}, 0\right),$$

with

$$\omega = \left(y^{(0)}, x^{(1)}, ..., x^{(d-1)}\right) \in \mathbb{S}_{d-1},\quad \omega' = \left({y'}^{(0)}, {x'}^{(1)}, ..., {x'}^{(d-1)}\right) \in \mathbb{S}_{d-1}.$$

The Minkowskian scalar product $z \cdot z'$ here reduces to $z \cdot z' = -\langle \omega \cdot \omega' \rangle$, where $\langle \omega \cdot \omega' \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$. Then $\langle \omega \cdot \omega' \rangle = \cos\Theta_t$ with Θ_t real.

We then intend to write the Bethe–Salpeter equation (2.2) in the Euclidean subspace E_{d+1} in terms of the radial, longitudinal and angular variables. We shall use the fact that the four-point function $H\left([k]\right)$ of the scalar fields $\left(\phi_1,\phi_2\right)$, and therefore also $B\left([k]\right)$ are invariant under the connected part of the complex Lorentz group $SO_0^{(c)}\left(1,d\right)$. This implies that these functions only depend on the Lorentz-invariant variables $\left(\rho,w,\rho',w',t\right)$, or equivalently $\left(\zeta,\zeta',t\right)$ and $z\cdot z'=-\left\langle\omega\cdot\omega'\right\rangle=-\cos\Theta_t$, the restrictions of these functions to the Euclidean subspace $\left(E_{d+1}\right)^3$ being then invariant under the connected orthogonal group $SO_0\left(d+1\right)$. One can thus put (with notations similar to those of [1], Eq. (3.24)):

$$H\left([k]\right)\equiv H\left[t;\rho,w;\rho',w';\left\langle\omega\cdot\omega'\right\rangle\right]\equiv\underline{H}_{\left(\zeta,\zeta',t\right)}\left(\left\langle\omega\cdot\omega'\right\rangle\right) \tag{3.7}$$

$$B\left([k]\right)\equiv B\left[t;\rho,w;\rho',w';\left\langle\omega\cdot\omega'\right\rangle\right]\equiv\underline{B}_{\left(\zeta,\zeta',t\right)}\left(\left\langle\omega\cdot\omega'\right\rangle\right) \tag{3.8}$$

We note that in view of Eq(3.1), the bounds (2.1) on H in $\left(E_{d+1}\right)^3$ can be rewritten as follows , with a suitable constant $C_E^{(H)}$:

$$\left|H\left[t;\rho,w;\rho',w';\left\langle\omega\cdot\omega'\right\rangle\right]\right| \;\leq$$

$$C_E^{(H)}(1+|t|^{\frac{1}{2}})^{N_H}\;(1+\rho)^{N_H}\;(1+|w||t|^{\frac{1}{2}})^{N_H}\;(1+\rho')^{N_H}\;(1+|w'||t|^{\frac{1}{2}})^{N_H}. \tag{3.9}$$

In view of the Lorentz invariance of the propagator $\Pi_i^{(\mathrm{reg})}\left(k\right)\equiv\Pi_i^{(\mathrm{reg})}\left[k^2\right]$ of $\phi_i\left(i=1,2\right)$, Eq. (2.3) can also be rewritten as follows:

$$G\left(K,Z\right)=\underline{G}\left(\zeta\right)=i\Pi_1^{(\mathrm{reg})}\left[\zeta_1\right]\Pi_2^{(\mathrm{reg})}\left[\zeta_2\right] \tag{3.10}$$

or

$$G\left[t;\rho,w\right]=i\Pi_1^{(\mathrm{reg})}\left[\left(w+1/2\right)^2t-\rho^2\right]\Pi_2^{(\mathrm{reg})}\left[\left(w-1/2\right)^2t-\rho^2\right], \tag{3.11}$$

and the bounds (2.4) then yield correspondingly:

$$|G[t;\rho,w]|\leq c^2[1+\rho^2+|t|(w+\frac{1}{2})^2]^{-r}[1+\rho^2+|t|(w-\frac{1}{2})^2]^{-r}. \tag{3.12}$$

In view of (3.1) ...(3.6), the integration measure dZ on E_{d+1} reads:

$$dZ=i\sqrt{-t}\rho^{d-1}d\rho\,dw\,d\omega=-id\mu_t\left(\zeta\right)d\omega \tag{3.13}$$

with

$$d\mu_t\left(\zeta\right)=\frac{1}{4\sqrt{-t}}\left(\frac{\Lambda\left(\zeta_1,\zeta_2,t\right)}{4t}\right)^{\frac{d-2}{2}}d\zeta_1,d\zeta_2 \tag{3.14}$$

We can now give the following alternative form of the BS-equation (2.2) in Euclidean space in terms of mass variables and angular variables; as shown in the Appendix, the bounds (3.9), (3.12) on H and G ensure that this integral relation is a genuine Fredholm equation depending on the parameter t for all $t<0$. One obtains:

$$\begin{aligned} H\left[t;\rho,w;\rho',w';\langle\omega\cdot\omega'\rangle\right] &= B\left[t;\rho,w;\rho',w';\langle\omega\cdot\omega'\rangle\right] \\ &\quad + \left(B\circ_t H\right)\left[t;\rho,w;\rho',w';\langle\omega\cdot\omega'\rangle\right], \end{aligned} \tag{3.15}$$

where:

$$\begin{aligned} \left(B\circ_t H\right)\left[t;\rho,w;\rho',w';\langle\omega\cdot\omega'\rangle\right] &= i\sqrt{-t}\int_0^\infty \rho''^{d-1}d\rho''\int_{-\infty}^{\infty} dw''\, G\left[t;\rho'',w''\right]\int_{\mathbb{S}_{d-1}} d\omega''\, B\left[t;\rho,w;\rho'',w'';\langle\omega\cdot\omega''\rangle\right] \\ &\quad \times H\left[t;\rho'',w'';\rho',w';\langle\omega''\cdot\omega'\rangle\right] \end{aligned} \tag{3.16}$$

or equivalently, by using the mass variables and introducing the convolution product $*$ of $SO\left(d\right)$-invariant kernels a and b on the sphere $\mathbb{S}_{d-1}$, namely

$$\left(a*b\right)\left(\langle\omega\cdot\omega'\rangle\right)=\int_{\mathbb{S}_{d-1}} d\omega''\, a\left(\langle\omega\cdot\omega''\rangle\right) b\left(\langle\omega''\cdot\omega'\rangle\right), \tag{3.17}$$

$$\underline{H}_{(\zeta,\zeta',t)}\left(\langle\omega\cdot\omega'\rangle\right)=\underline{B}_{(\zeta,\zeta',t)}\left(\langle\omega\cdot\omega'\rangle\right) - i\int_{\Delta_t}\left(\underline{B}_{(\zeta,\zeta'',t)} * \underline{H}_{(\zeta'',\zeta',t)}\right)\left(\langle\omega\cdot\omega'\rangle\right)\underline{G}\left(\zeta''\right)d\mu_t\left(\zeta''\right). \tag{3.18}$$

Let us now introduce the "partial-wave expansion" of invariant kernels $a\left(\langle\omega\cdot\omega'\rangle\right)\equiv a\left(\cos\theta\right)$ on the sphere $\mathbb{S}_{d-1}$ by the following formulae:

$$a\left(\cos\theta\right)=\frac{1}{\omega_d}\sum_{0\le\ell<\infty}\tilde{a}_\ell\ \mathrm{h}_d\left(\ell\right)P_\ell^{(d)}\left(\cos\theta\right), \tag{3.19}$$

$$\tilde{a}_\ell=\omega_{d-1}\int_{-1}^{+1}P_\ell^{(d)}\left(\cos\theta\right)a\left(\cos\theta\right)\left(\sin\theta\right)^{d-3}d\cos\theta \tag{3.20}$$

where the functions $P_\ell^{(d)}$ are the "ultraspherical Legendre polynomials", given by the following integral representation:

$$P_\ell^{(d)}\left(\cos\theta\right)=\frac{\omega_{d-2}}{\omega_{d-1}}\int_0^\pi\left(\cos\theta+i\sin\theta\cos\varphi\right)^\ell\left(\sin\varphi\right)^{d-3}d\varphi, \tag{3.21}$$

$$\mathrm{h}_d\left(\lambda\right)=\frac{\left(2\lambda+d-2\right)}{\left(d-2\right)!}\frac{\Gamma\left(\lambda+d-2\right)}{\Gamma\left(\lambda+1\right)} \tag{3.22}$$

and $\omega_d=2\frac{\pi^{d/2}}{\Gamma(d/2)}$ is the area of the sphere $\mathbb{S}_{d-1}$.

Eq. (3.20) allows us to introduce the "$t-$channel partial-waves" $\tilde{h}_\ell[t;\rho,w,\rho',w']\equiv\tilde{h}_\ell(\zeta,\zeta',t)$ of the restriction of the four-point function $H\left([k]\right)$ to E^3_{d+1} :

$$\tilde{h}_\ell\left(\zeta,\zeta',t\right)=\omega_{d-1}\int_{-1}^{+1}P_\ell^{(d)}\left(\cos\theta\right)\underline{H}_{(\zeta,\zeta',t)}\left(\cos\theta\right)\left(\sin\theta\right)^{d-3}d\cos\theta, \tag{3.23}$$

and similarly for $B\left([k]\right)$:

$$\tilde{b}_\ell\left(\zeta,\zeta',t\right)=\omega_{d-1}\int_{-1}^{+1}P_\ell^{(d)}\left(\cos\theta\right)\underline{B}_{(\zeta,\zeta',t)}\left(\cos\theta\right)\left(\sin\theta\right)^{d-3}d\cos\theta. \tag{3.24}$$

By now using the "factorization property"[2] according to which the partial waves of $(a * b)(\langle \omega \cdot \omega' \rangle)$ are:

$$(\widetilde{a * b})_\ell = \widetilde{a}_\ell \widetilde{b}_\ell, \tag{3.25}$$

we can replace the version (3.18) of the Bethe–Salpeter equation by the following set of "*Bethe–Salpeter equations for the partial waves*" of $H([k])$:

$$\tilde{h}_\ell(\zeta, \zeta', t) = \tilde{b}_\ell(\zeta, \zeta', t) + \int_{\Delta_t} \tilde{b}_\ell(\zeta, \zeta''; t)\, \tilde{h}_\ell(\zeta'', \zeta'; t)\, \underline{G}(\zeta'')\, d\mu_t(\zeta'')\,. \tag{3.26}$$

Note that each of these integral equations is comparable to a Bethe–Salpeter equation for field theory in two-dimensional space-time. In view of the bounds (3.9) for H and (3.12) for G, the integral equations (3.15) and (3.26) appear as Fredholm resolvent equations, to which the results of the Appendix apply (see Proposition A1 with z, z' varying in Γ_0 for Eq (3.15) and the remark after Proposition A3 for Eqs (3.26)).

3.2 Bethe–Salpeter equation for absorptive parts in the mass variables and angular variables: perikernel structure

In section 2, we have recalled the fact that the axiomatic analyticity domain of $H([k])$ (or $B([k])$) contains the Euclidean subspace of complex momentum-space and provides a connection between the latter and the *real* Minkowskian subspace by travelling in the complex energy variables $K^{(0)}, Z^{(0)}, Z'^{(0)}$ at fixed $\vec{K}, \vec{Z}, \vec{Z}'$. This allowed one to reach the spectral sets Σ_s, Σ_u, to compute the corresponding discontinuities of $B \circ_t H$ and thereby to obtain the Bethe–Salpeter equations (2.6) and (2.7) for the absorptive parts of $H([k])$ in the s and u-channels. This derivation was valid for arbitrary vectors $K = (K^{(0)}$ complex, $\vec{K}$ real).

For $K = (0, \vec{K})(K^2 = t < 0)$, an analyticity property of similar type, but actually different since *adapted to the mass variables and angular variables* $(\zeta, z), (\zeta', z')$, was established in [1].In fact, it was proven there that for all (real) values of (ζ, ζ') in $\Delta_t \times \Delta_t$, the "*enlarged*" axiomatic analyticity domain of $H([k])$, obtained by geometrical techniques of analytic completion, contains the whole complex manifold $\hat{\Omega}_{(\zeta, \zeta', K)}$ parametrized by Eqs. (3.1) ...(3.6), in which (z, z') varies on $X_{d-1}^{(c)} \times X_{d-1}^{(c)}$, deprived from "cuts" generated by the spectral sets Σ_s $((Z - Z')^2 \geq s_0)$ and Σ_u $((Z + Z')^2 \geq u_0)$. In other words, for each (ζ, ζ') fixed, the function $\underline{H}_{(\zeta, \zeta', t)}(-z \cdot z') \equiv H([k])$ considered in subsection 3.1 as an invariant kernel on the sphere $\mathbb{S}_{d-1}$ admits *an analytic continuation on* $X_{d-1}^{(c)} \times X_{d-1}^{(c)}$ deprived from cuts $\underline{\Sigma}_s(\zeta, \zeta', t)$ and $\underline{\Sigma}_u(\zeta, \zeta', t)$ (see Eqs. (3.29), (3.30) below) which describe the traces of the spectral sets Σ_s

[2]Note that the normalizations chosen for writing the definitions (3.17) and (3.20) of the convolution-product and of the partial waves yield Eq. (3.25) without extra-coefficient; they however differ by a factor ω_d from the standard normalization.

and Σ_u in the manifold $\hat{\Omega}_{(\zeta,\zeta',K)}$. Moreover, since $\underline{H}_{(\zeta,\zeta',t)}$ only depends on z, z' through the variable $\cos\Theta = -z\cdot z'$, it is analytic with respect to this variable in the image of $X_{d-1}^{(c)} \times X_{d-1}^{(c)} \backslash (\underline{\Sigma}_s(\zeta,\zeta',t) \cup \underline{\Sigma}_u(\zeta,\zeta',t))$, which is a *cut-plane* of the form $\mathbb{C}\backslash \{[\cosh v_s + \infty[\cup] - \infty, -\cosh v_u]\}$.

The class of functions $\mathcal{K}(z,z')$ holomorphic in the previous cut-domains of $X_{d-1}^{(c)} \times X_{d-1}^{(c)}$ (also denoted by $\mathcal{K}(-z\cdot z')$ when they are Lorentz invariant) has been extensively studied in [11,12] under the name of *(invariant) perikernels,* and their discontinuities have been characterized as *(invariant)* "*Volterra kernels*" on the one-sheeted hyperboloid X_{d-1}; useful results involving these notions will be recalled below.

Such a perikernel structure is satisfied not only by $H([k])$ but also by the BS-kernel $B([k])$, since the latter enjoys the same analyticity domain as $H([k])$ for fixed K (with $t = K^2 < 0$); so we shall put similarly $B([k]) \equiv \underline{B}_{(\zeta,\zeta',t)}(-z\cdot z')$. This perikernel structure of $\underline{B}_{(\zeta,\zeta',t)}(-z\cdot z')$ will also be reobtained below in two ways by introducing and solving appropriate extensions of the BS-equations (3.18) and (3.26).

In the continuation of this program, a basic role is played by the bounds (2.1) which $H([k])$ is assumed to satisfy in its axiomatic domain, in particular in the sets $\underline{D}_{(w,w',\rho')}$ of Propositions 3, 4 of [1]. In fact, it has been established in theorem 1 of [1] that bounds of the following form hold in each submanifold $\hat{\Omega}_{(\zeta,\zeta',K)}$:

$$|\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)| \leq C_{(\zeta,\zeta',t)}^{(H)}\, e^{N_H|\Im m\Theta_t|}|\sin\Re e\Theta_t|^{-n_H}. \tag{3.27}$$

In the latter, n_H describes the maximal local order of singularity of the distribution-boundary-values of $\underline{H}_{(\zeta,\zeta',t)}$ on the reals. We shall consider here n_H as being independent [3] of the exponent N_H which governs the behaviour of H at infinity according to the assumed bounds (2.1). Concerning the "constant" $C_{(\zeta,\zeta',t)}^{(H)}$, one can check (by following the proof of theorem 1 of [1]) that it can be taken equal to the uniform bound of H in the Euclidean subspace, namely (see (3.9)):

$$C_{(\zeta,\zeta',t)}^{(H)} = C_E^{(H)}[(1+|t|^{\frac{1}{2}})\,(1+\rho)\,(1+|w||t|^{\frac{1}{2}})\,(1+\rho')\,(1+|w'||t|^{\frac{1}{2}})]^{N_H}. \tag{3.28}$$

Absorptive parts: passage to the mass variables and hyperbolic angular variables; convolution products $\lozenge$

The absorptive parts $\Delta_s H, \Delta_u H$ (resp. $\Delta_s B, \Delta_u B$) defined in section 2 also appear at fixed (ζ,ζ',t) (or (ρ,w,ρ',w',t)) as being equal to the discontinuities $\Delta_s\underline{H}_{(\zeta,\zeta',t)}(-z\cdot z')$ and $\Delta_u\underline{H}_{(\zeta,\zeta',t)}(-z\cdot z')$ of $\underline{H}_{(\zeta,\zeta',t)}(-z\cdot z')$ (and similarly for $\underline{B}_{(\zeta,\zeta',t)}$). The supports of these discontinuities, obtained by writing the conditions $s = (Z-Z')^2 \geq s_0, u = (Z+Z')^2 \geq u_0$ in terms of the parametrization (3.1), (3.2) are respectively:

$$\underline{\Sigma}_s(\zeta,\zeta',t) = \{(z,z') \in X_{d-1}^{(c)} \times X_{d-1}^{(c)};\ -z\cdot z' = \cosh v \geq \cosh v_s\},$$

[3] The occurrence of $\max(n_H, N_H)$ in place of N_H in the exponential factor obtained in the bound (3.25) of [1] is in fact without physical content

$$\text{with } \cosh v_s = 1 + \frac{s_0 + (\rho - \rho')^2 - (w - w')^2\, t}{2\rho\rho'} \tag{3.29}$$
$$(\text{or } \Theta_t = iv, \qquad v \geq v_s);$$

$$\underline{\Sigma}_u\left(\zeta, \zeta', t\right) = \ \{(z, z') \in X_{d-1}^{(c)} \times X_{d-1}^{(c)};\ z \cdot z' = \cosh v \geq \cosh v_u\},$$

$$\text{with } \cosh v_u = 1 + \frac{u_0 + (\rho - \rho')^2 - (w + w')^2\, t}{2\rho\rho'} \tag{3.30}$$
$$(\text{or } \Theta_t = \pi + iv, \qquad v \geq v_u).$$

It is easy to check that these sets are respectively contained in the regions $(z - z')^2 > 0$ and $(z + z')^2 > 0$. For z, z' real, i.e. belonging to the real one-sheeted hyperboloid X_{d-1}, these relations express the fact that z is either in the "future" or in the "past" of z' (resp.$-z'$). (Note that the future and the past of z', namely the regions $\Gamma^{\pm}(z') = X_{d-1} \cap \left\{z \in \mathbb{R}^{d-1}, z - z' \in V^{\pm}\right\}$ are bounded by the cone of generatrices of X_{d-1} passing through z'). We conclude that the trace of $\underline{\Sigma}_s\left(\zeta, \zeta', t\right)$ on $X_{d-1} \times X_{d-1}$ is composed of two disjoint sets $\underline{\Sigma}_s^+\left(\zeta, \zeta', t\right)$ and $\underline{\Sigma}_s^-\left(\zeta, \zeta', t\right)$, corresponding respectively to the conditions $z - z' \in \bar{V}^+$ and $z - z' \in \bar{V}^-$. Of course, these sets represent the supports of the respective components $\Delta_s^+ H$ and $\Delta_s^- H$ of $\Delta_s H$ in the manifold $\hat{\Omega}_{(\zeta, \zeta', K)}$, and this leads one to distinguish two kernels $\Delta_s^+ \underline{H}_{(\zeta,\zeta',t)}(z, z') \equiv \Delta_s^+ H([k])$ and $\Delta_s^- H_{(\zeta,\zeta',t)}(z, z') \equiv \Delta_s^- H([k])$ of disjoint supports on X_{d-1}, although (in view of Lorentz invariance) they are both represented *by the same function of one variable* denoted earlier by $\Delta_s \underline{H}_{(\zeta,\zeta',t)}(-z \cdot z')$. Kernels on X_{d-1} such as $\Delta_s^+ \underline{H}_{(\zeta,\zeta',t)}(z, z')$, whose support is contained in the set $\left\{(z, z') \in X_{d-1} \times X_{d-1}; z - z' \in \bar{V}^+\right\}$ have been introduced in [14] under the name of "*Volterra kernels*" on X_{d-1}. One introduces similarly the kernels $\Delta_u^{\pm} \underline{H}_{(\zeta,\zeta',t)}(z, z')$ with respective supports $\underline{\Sigma}_u^{\pm}\left(\zeta, \zeta', t\right)$ distinguished by the conditions $\pm(z + z') \in \bar{V}^+$, which are represented by the same function $\Delta_u \underline{H}_{(\zeta,\zeta',t)}(-z \cdot z')$.

These considerations will now allow us to reinterpret the Bethe–Salpeter equations for the absorptive parts (2.6) and (2.7), after rewriting the latter in terms of the mass variables (ζ, ζ') and of the "hyperbolic angular variables" z, z', varying on X_{d-1}. As a counterpart of our expression (3.16) of $(B \circ_t H)$ in Euclidean space, we can in fact rewrite the corresponding composition product $\Delta_s B \lozenge_t \Delta_s H$ (see Eq. (2.10)) as follows, in the situation where Z is in the future of Z' (a similar expression would be obtained in the opposite situation):

$$(\Delta_s B \lozenge_t \Delta_s H)\,[t; \rho, w; \rho', w'; -z \cdot z'] =$$

$$\sqrt{-t} \int_0^{\infty} \rho''^{d-1} d\rho'' \int_{-\infty}^{+\infty} G\,[t; \rho'', w'']\, dw'' \int\limits_{\lozenge(z,z') \cap X_{d-1}} \Delta_s B\,[t; \rho, w; \rho'', w''; -z \cdot z''] \times$$

$$\cdots \Delta_s H\left[t;\rho'',w'';\rho',w';-z''\cdot z'\right]dz'', \tag{3.31}$$

where dz'' denotes the Lorentz-invariant measure $dz'' = \dfrac{dz_0 ... dz_{d-2}}{z_{d-1}}$ on X_{d-1}. The fact that the integration region on X_{d-1} is restricted to the double-cone $\Diamond(z,z')$ is a consequence of the integration prescription $\{Z'' \in \Diamond(Z,Z')\}$ expressing the support properties of $\Delta_s B(K;Z,Z')$ and $\Delta_s H(K;Z,Z')$ in Eq. (2.10). This results from the implication relation $Z - Z' \in \overline{V}^+ \Longrightarrow z - z' \in \bar{V}^+$, obvious from the following identities (entailed by Eqs. (3.1), (3.2)):

$$\rho\rho'\left(z-z'\right)^2 = \left(Z-Z'\right)^2 + \left(\rho-\rho'\right)^2 + \left(w-w'\right)^2|t| \tag{3.32}$$

and

$$\left(Z-Z'\right)\cdot\left(z-z'\right) = \left(\rho+\rho'\right)\times\frac{\left(z-z'\right)^2}{2} \tag{3.33}$$

Moreover the same support properties of $\Delta_s B$ and $\Delta_s H$ also imply that the integrand at the r.h.s. of Eq.(3.31) vanishes outside a compact subset of the space of integration variables (ρ'',w''); the integral (3.31) is therefore well-defined independently of the bounds on $\Delta_s B$ and $\Delta_s H$.

It is now appropriate to introduce the notion of *convolution product* $\Diamond$ of Volterra kernels on the one-sheeted hyperboloid (see [14,11]) by the following formula:

$$\left(F_1 \Diamond F_2\right)(z,z') = \int\limits_{\Diamond(z,z')\cap X_{d-1}} F_1\left(z,z''\right)F_2\left(z'',z'\right)dz''. \tag{3.34}$$

It is to be noted that, due to the compactness of the integration region in (3.34), this convolution product remains meaningful for distribution-like Volterra kernels.

For invariant Volterra kernels, which are of the form $F_i(z,z') = f_i(-z\cdot z')$, $i=1,2$ (as it is the case here for $\Delta_s^+\underline{H}_{(\zeta,\zeta',t)}, \Delta_s^+\underline{B}_{(\zeta,\zeta',t)}$), formula (3.34) takes the following alternative form:

$$\left(F_1 \Diamond F_2\right)(z,z') \equiv \left(f_1 \blacklozenge f_2\right)(\cosh v) = \frac{2\omega_{d-2}}{\sinh v^{d-3}} \int\limits_{\substack{v_1\geq 0, v_2 \geq 0\\ v_1+v_2\leq v}} f_1\left(\cosh v_1\right) f_2\left(\cosh v_2\right)$$

$$\cdots\left[\left(\cosh v - \cosh\left(v_1+v_2\right)\right)\left(\cosh v - \cosh\left(v_1-v_2\right)\right)\right]^{\frac{d-4}{2}} d\left(\cosh v_1\right) d\left(\cosh v_2\right). \tag{3.35}$$

Using the latter and putting $\Delta_s H\left[t;\rho,w;\rho',w',-z\cdot z'\right] \equiv \Delta_s \underline{H}_{(\zeta,\zeta',t)}\left(-z\cdot z'\right)$ (and similarly for $\Delta_s B$), we can then rewrite Eq. (3.31) as follows:

$$\left(\Delta_s B \ \Diamond_t \ \Delta_s H\right)\left[t;\rho,w;\rho',w';-z\cdot z'\right] =$$

$$\int\limits_{\Delta_t} \left(\Delta_s \underline{B}_{(\zeta,\zeta'',t)} \blacklozenge \Delta_s \underline{H}_{(\zeta'',\zeta',t)}\right)\left(-z\cdot z'\right) \underline{G}\left(\zeta''\right) d\mu_t\left(\zeta''\right). \tag{3.36}$$

Similar expressions could be written for the three other composition products $\Delta_u B \ \Diamond_t \ \Delta_u H, \ \Delta_s B \ \Diamond_t \ \Delta_u H, \ \Delta_u B \ \Diamond_t \ \Delta_s H$ of Eqs. (2.6), (2.7), in terms of corresponding convolution products of Volterra kernels.

From the Euclidean BS-equation to the BS-equation for s and $u-$channel absorptive parts through contour-distortion of "perikernel convolution products"

We now recall the basic relationship which relates the $*$-convolution product of kernels on the "Euclidean" sphere $\mathbb{S}_{d-1}$ and the $\Diamond-$convolution product of Volterra kernels on the one-sheeted hyperboloid X_{d-1} (theorem 2' of [11]).

Being given two perikernels $\mathcal{K}_i\left(z,z'\right), i = 1,2,$ on $X_{d-1}^{(c)}$ whose respective discontinuities on the sets $\left\{z-z' \in V^{\pm}\right\}, \left\{z+z' \in V^{\pm}\right\}$ are the Volterra kernels $\Delta_s^{\pm}\mathcal{K}_i\left(z,z'\right), \Delta_u^{\pm}\mathcal{K}_i\left(z,z'\right)$, there exists a perikernel $\mathcal{K}$ denoted by $\mathcal{K} = \mathcal{K}_1 *^{(c)} \mathcal{K}_2$, such that:

i) the restrictions of $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$, to the "Euclidean" sphere $\mathbb{S}_{d-1}$ are such that:

$$\mathcal{K}_{|\mathbb{S}_{d-1}} = \mathcal{K}_{1|\mathbb{S}_{d-1}} * \mathcal{K}_{2|\mathbb{S}_{d-1}}, \tag{3.37}$$

ii) the discontinuities $\Delta_s^{+}\mathcal{K}\left(z,z'\right)$ and $\Delta_u^{+}\mathcal{K}\left(z,z'\right)$ are given by the following $\Diamond-$convolution products:

$$\Delta_s^{+}\mathcal{K} = \Delta_s^{+}\mathcal{K}_1 \Diamond \Delta_s^{+}\mathcal{K}_2 + \Delta_u^{-}\mathcal{K}_1 \Diamond \Delta_u^{+}\mathcal{K}_2$$

$$\Delta_u^{+}\mathcal{K} = \Delta_s^{-}\mathcal{K}_1 \Diamond \Delta_u^{+}\mathcal{K}_2 + \Delta_u^{+}\mathcal{K}_1 \Diamond \Delta_s^{+}\mathcal{K}_2, \tag{3.38}$$

(similar formulae being satisfied by $\Delta_s^{-}\mathcal{K}$ and $\Delta_u^{-}\mathcal{K}$),

iii) for every (z,z') in the analyticity domain of $\mathcal{K}$, there exists a class of cycles $\Gamma(z,z')$ such that

$$\mathcal{K}(z,z') = \int_{\Gamma(z,z')} \mathcal{K}_1(z,z'')\mathcal{K}_2(z'',z')dz'', \tag{3.39}$$

$\Gamma(z,z')$ being obtained by continuous distortion inside the analyticity domain of the integrand from the special cycle $\Gamma_0(z,z') \equiv \mathbb{S}_{d-1}$, relevant for the Euclidean configurations $(z,z') \in \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$,

iv) if $\mathcal{K}_1, \mathcal{K}_2$ are invariant perikernels, $\mathcal{K}$ is also an invariant perikernel.

This statement can then be applied for each set of fixed values of $\left(\zeta,\zeta',\zeta'',t\right)$ to the (invariant) perikernels $\underline{B}_{(\zeta,\zeta'',t)}(-z\cdot z'')$ and $\underline{H}_{(\zeta'',\zeta',t)}(-z''\cdot z')$. It implies the existence of an (invariant) perikernel $\left(\underline{B}_{(\zeta,\zeta'',t)} *^{(c)} \underline{H}_{(\zeta'',\zeta',t)}\right)(-z\cdot z')$, whose restriction to the "Euclidean" sphere $\mathbb{S}_{d-1}$ is correspondingly the kernel $\left(\underline{B}_{(\zeta,\zeta'',t)} * \underline{H}_{(\zeta'',\zeta',t)}\right)$ appearing at the r.h.s. of Eq. (3.18).

In view of property iii), one can see that *the Bethe–Salpeter equation (3.18) can be analytically continued* to all (z,z') in $X_{d-1}^{(c)} \times X_{d-1}^{(c)}$, for all $t<0$, $\left(\zeta,\zeta'\right)$

in $\Delta_t \times \Delta_t$, under the following form:

$$\underline{H}_{(\zeta,\zeta',t)}(z,z') = \underline{B}_{(\zeta,\zeta',t)}(z,z') - i \int\limits_{\Delta_t} \left(\underline{B}_{(\zeta,\zeta'',t)} *^{(c)} \underline{H}_{(\zeta'',\zeta',t)} \right)(z,z')\, \underline{G}(\zeta'')\, d\mu_t(\zeta''). \tag{3.40}$$

The latter can be considered as a Fredholm resolvent equation in complex space whose integration space is the product of Δ_t by the "floating cycle" Γ on which the complex points z, z' vary. In view of general results of [13] adapted to the present situation in the Appendix, *the function $\underline{B}_{(\zeta,\zeta',t)}(z,z')$ is directly obtained with its full perikernel structure as the solution of the Fredholm equation (3.40)*: in fact, $\underline{B}_{(\zeta,\zeta',t)}(z,z')$ can be identified with the resolvent $R_{H|\alpha=-1}(t;\rho,w,\rho',w',z,z')$ of Propositions A1 and A2 (F being replaced by H). In the case when the boundary values of $\underline{H}_{(\zeta,\zeta',t)}(z,z')$ and its discontinuities are continuous (namely if the bounds (3.27) hold with $n_H = 0$), the Fredholm solution $\underline{B}_{(\zeta,\zeta',t)}(z,z')$ of Eq. (3.40) satisfies the same regularity properties and therefore one can apply the property ii) (Eq. (3.38)) of the $*^{(c)}-$composition product of perikernels for computing side-by-side the discontinuities of Eq. (3.40). This yields:

$$\Delta_s^+ \underline{H}_{(\zeta,\zeta',t)} = \Delta_s^+ \underline{B}_{(\zeta,\zeta',t)} + \tag{3.41}$$

$$\int_{\Delta_t} \left[\Delta_s^+ \underline{B}_{(\zeta,\zeta'',t)} \Diamond \Delta_s^+ \underline{H}_{(\zeta'',\zeta',t)} + \Delta_u^- \underline{B}_{(\zeta,\zeta'',t)} \Diamond \Delta_u^+ \underline{H}_{(\zeta'',\zeta',t)} \right] \underline{G}(\zeta'')\, d\mu_t(\zeta'')$$

$$\Delta_u^+ \underline{H}_{(\zeta,\zeta',t)} = \Delta_u^+ \underline{B}_{(\zeta,\zeta',t)} + \tag{3.42}$$

$$\int_{\Delta_t} \left[\Delta_s^- \underline{B}_{(\zeta,\zeta'',t)} \Diamond \Delta_u^+ \underline{H}_{(\zeta'',\zeta',t)} + \Delta_u^+ \underline{B}_{(\zeta,\zeta'',t)} \Diamond \Delta_s^+ \underline{H}_{(\zeta'',\zeta',t)} \right] \underline{G}(\zeta'')\, d\mu_t(\zeta'')$$

(with similar expressions for $\Delta_{s,u}^- \underline{H}_{(\zeta,\zeta',t)}(z,z')$). Then by comparing the convolution products here obtained with the forms (3.31),(3.36) of the $\Diamond_t-$composition-product (2.10), we see that the latter equations are in fact identical to Eqs (2.6),(2.7). So in this new presentation making use of perikernel-convolution-products, we have reobtained the BS-equations for the absorptive parts in terms of mass variables and hyperbolic angular variables.

We now make use of the fact that both Volterra kernels $\left(\Delta_s^+ \underline{H}_{(\bullet)}, \Delta_s^- \underline{H}_{(\bullet)}\right)$ are represented by the same Lorentz-invariant function $\Delta_s \underline{H}_{(\bullet)}(-z\cdot z')$, with $-z\cdot z' = \cosh v \geq 1$, while both Volterra kernels $\left(\Delta_u^+ \underline{H}_{(\bullet)}, \Delta_u^- \underline{H}_{(\bullet)}\right)$ are represented by the same function $\Delta_u \underline{H}_{(\bullet)}(-z\cdot z')$, with $-z\cdot z' = -\cosh v \leq -1$. It is then convenient to put $\widehat{\Delta_u \underline{H}}_{(\bullet)}(\cosh v) = \Delta_u \underline{H}_{(\bullet)}(-\cosh v)$; similar considerations are done for $\underline{B}_{(\bullet)}$. Then, Eqs. (3.41), (3.42) can be rewritten in terms of $\blacklozenge-$convolution products of invariant Volterra kernels (see Eq.(3.35)) as follows:

$$\Delta_s \underline{H}_{(\zeta,\zeta',t)} = \Delta_s \underline{B}_{(\zeta,\zeta',t)} + \cdots \tag{3.43}$$

$$\Delta_s \underline{B}_{(\zeta,\zeta'',t)} \blacklozenge \Delta_s \underline{H}_{(\zeta'',\zeta',t)} + \widehat{\Delta_u \underline{B}}_{(\zeta,\zeta'',t)} \blacklozenge \widehat{\Delta_u \underline{H}}_{(\zeta'',\zeta',t)}$$

and

$$\widehat{\Delta_u \underline{H}}_{(\zeta,\zeta',t)} = \widehat{\Delta_u \underline{B}}_{(\zeta,\zeta',t)} + \cdots \tag{3.44}$$

$$\Delta_s \underline{B}_{(\zeta,\zeta'',t)} \blacklozenge \widehat{\Delta_u \underline{H}}_{(\zeta'',\zeta',t)} + \widehat{\Delta_u \underline{B}}_{(\zeta,\zeta'',t)} \blacklozenge \Delta_s \underline{H}_{(\zeta'',\zeta',t)}.$$

In these equations , all the terms are functions of the variable $\cosh v$ varying on the half-line $[1, +\infty[$.

Symmetrized and antisymmetrized Bethe–Salpeter equations for the absorptive parts:

Let us put

$$\Delta^{(s)} \underline{B}_{(\bullet)} = \Delta_s \underline{B}_{(\bullet)} + \widehat{\Delta_u \underline{B}}_{(\bullet)}, \quad \Delta^{(s)} \underline{H}_{(\bullet)} = \Delta_s \underline{H}_{(\bullet)} + \widehat{\Delta_u \underline{H}}_{(\bullet)}, \tag{3.45}$$

and

$$\Delta^{(a)} \underline{B}_{(\bullet)} = \Delta_s \underline{B}_{(\bullet)} - \widehat{\Delta_u \underline{B}}_{(\bullet)}, \quad \Delta^{(a)} \underline{H}_{(\bullet)} = \Delta_s \underline{H}_{(\bullet)} - \widehat{\Delta_u \underline{H}}_{(\bullet)}, \tag{3.46}$$

Then by adding-up and subtracting Eqs. (3.43) and (3.44) side by side one obtains the following *decoupled Bethe–Salpeter equations for the symmetric and antisymmetric combinations of the s and u-channel absorptive parts*:

$$\Delta^{(s)} \underline{H}_{(\zeta,\zeta',t)} = \Delta^{(s)} \underline{B}_{(\zeta,\zeta',t)} + \int\limits_{\Delta_t} \Delta^{(s)} \underline{B}_{(\zeta,\zeta'',t)} \blacklozenge \Delta^{(s)} \underline{H}_{(\zeta'',\zeta',t)} \underline{G}(\zeta'') d\mu_t \left(\zeta''\right) \tag{3.47}$$

and

$$\Delta^{(a)} \underline{H}_{(\zeta,\zeta',t)} = \Delta^{(a)} \underline{B}_{(\zeta,\zeta',t)} + \int\limits_{\Delta_t} \Delta^{(a)} \underline{B}_{(\zeta,\zeta'',t)} \blacklozenge \Delta^{(a)} \underline{H}_{(\zeta'',\zeta',t)} \underline{G}(\zeta'') d\mu_t \left(\zeta''\right). \tag{3.48}$$

The case of distribution-like boundary values

This more general case is characterized by a *positive* exponent n_H in the bound (3.27) on $\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$. Then the function $\underline{B}_{(\zeta,\zeta',t)}(\cos\Theta_t)$ obtained as the Fredholm resolvent of $\underline{H}_{(\zeta,\zeta',t)}$ from Eq. (3.40) cannot be proved directly to enjoy a similar power bound near the reals (in view of the occurrence of the factor $|\sin \Re e\Theta_t|^{-n_H}$ in the constant M_Γ inside the argument of the entire function Φ' in the bound (A.20)). The fact that such a power and the corresponding distribution character of the boundary values of $\underline{B}_{(\zeta,\zeta',t)}$ still hold true (near the s and $u-$cuts) is a consequence of the following properties:

a) For any Feynman convolution $F_1 \circ_t F_2$ and for any composition product of perikernels $\mathcal{K}_1 *^{(c)} \mathcal{K}_2$, the discontinuity formulae (2.12),(2.13) and (3.38) can still be justified (by the corresponding contour-distortion arguments) when the boundary values are not continuous but governed by power bounds near the reals. These formulae then hold as well-defined convolution-type products of distributions with supports in a salient cone (resulting in the double-cone-shaped integration region of the $\lozenge_t$ and $\lozenge-$products).

b) In view of the additivity property of spectral regions (see Sec 2), any four-point function of the form $H^{\circ_t(n+1)} = H \circ_t \cdots \circ_t H$ (n products) has absorptive parts $\Delta_s H^{\circ_t(n+1)}$ and $\Delta_u H^{\circ_t(n+1)}$ admitting thresholds s_n and u_n "of order n": to make it simple, in the case when $s_0 = u_0$ one has $s_n = u_n = (n+1)^2 s_0$.

c) For every n let $B_{n+1}(K, Z, Z')$ be the solution of the auxiliary equation $H^{\circ_t(n+1)} = B_{n+1} + (-1)^n B_{n+1} \circ_t H^{\circ_t(n+1)}$. B_{n+1} has the same analyticity domain as $H^{\circ_t(n+1)}$, namely it has the same thresholds $s = s_n$, $u = u_n$. This results either from the Fredholm-type analysis of [8,9] or from the present one in the perikernel framework by using contours Γ in $X_{d-1}^{(c)}$ (see [11] and the Appendix).

d) For every n the following relation is obtained by iterating the equation $B = H - B \circ_t H$ and taking into account the defining equation of B_{n+1}:

$$B = \sum_{p=1}^{n+1} (-1)^{p-1} H^{\circ_t p} + \sum_{p=1}^{n+1} (-1)^{n+p} B_{n+1} \circ_t H^{\circ_t p}. \tag{3.49}$$

Being interested in the boundary values and absorptive parts $\Delta_s B, \Delta_u B$ of B in the real regions $\mathcal{R}_s^{(n)}$, $\mathcal{R}_u^{(n)}$ where (respectively) $s = (Z - Z')^2 < s_n$ and $u = (Z + Z')^2 < u_n$, one can see that only the terms of the first sum at the r.h.s. of Eq. (3.49) will contribute to these absorptive parts. In fact in view of c) all the terms of the second sum in Eq. (3.49) are analytic in the regions $\mathcal{R}_s^{(n)}$ and $\mathcal{R}_u^{(n)}$. Therefore in view of a) the existence of power bounds near the reals for B and the formulae for the corresponding absorptive parts in $\mathcal{R}_s^{(n)}$ and $\mathcal{R}_u^{(n)}$ are completely governed by the finite sum $\sum_{p=1}^{n+1} (-1)^{p-1} H^{\circ_t p}$. It then also follows that one can apply a) directly to the Feynman convolution $B \circ_t H$ in these regions and therefore derive Eqs (3.41), (3.42) by applying the contour-distortion argument to Eq. (3.40) (as in the case of continuous boundary values).

3.3 Complex angular momentum diagonalization of the Bethe–Salpeter equation

We shall now apply the main result of [1] (theorem 5) to the four-point function $H([k])$. This result relies basically on the Laplace-type L_d-transformation (see [12] for a complete study) which associates with each invariant Volterra kernel with moderate growth $F(z, z') \equiv f(-z \cdot z')$ on X_{d-1} the following analytic function (see proposition III-3 of [12]):

$$\tilde{F}(\lambda) = \omega_{d-1} \int_0^\infty f(\cosh v) Q_\lambda^{(d)}(\cosh v)\ (\sinh v)^{d-2} dv. \tag{3.50}$$

In this equation, $Q^{(d)}_{\lambda}(\cosh v)$ denotes the second-kind Legendre function in dimension d whose integral representation is given by Eq.(4.36) of [1]. If $|f(\cosh v)|$ is majorized by cst e^{Nv}, then $\tilde{F}(\lambda)$ is proved to be holomorphic in the half-plane $\mathbb{C}^{(N)}_{+} = \{\lambda \in \mathbb{C};\ \Re e\lambda > N\}$.

According to theorem 5 of [1], the absorptive parts $\Delta_s \underline{H}_{(\zeta,\zeta',t)}$ and $\Delta_u \underline{H}_{(\zeta,\zeta',t)}$ of a function $\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$ satisfying the bounds (3.27) admit "*t*-channel Laplace-type transforms" $\tilde{H}_s\left(\zeta,\zeta';t,\lambda_t\right)$ and $\tilde{H}_u\left(\zeta,\zeta';t,\lambda_t\right)$, which are holomorphic with respect to the *complex angular momentum variable* λ_t in the half-plane $\mathbb{C}^{(N_H)}_{+}$. These Laplace-type transforms, obtained by applying the L_d-transformation (3.50) at fixed values of ζ,ζ',t are:

$$\tilde{H}_s\left(\zeta,\zeta';t,\lambda_t\right) = \omega_{d-1}\int_0^{\infty} \Delta_s \underline{H}_{(\zeta,\zeta',t)}\left(\cosh v\right) Q^{(d)}_{\lambda_t}\left(\cosh v\right)\left(\sinh v\right)^{d-2} dv \tag{3.51}$$

$$\tilde{H}_u\left(\zeta,\zeta';t,\lambda_t\right) = \omega_{d-1}\int_0^{\infty} \Delta_u \underline{H}_{(\zeta,\zeta',t)}\left(-\cosh v\right) Q^{(d)}_{\lambda_t}\left(\cosh v\right)\left(\sinh v\right)^{d-2} dv \tag{3.52}$$

In the general case, these formulae have to be understood in the sense of distributions, namely with $Q^{(d)}_{\lambda_t}\left(\cosh v\right)\left(\sinh v\right)^{d-2}$ playing the role of a test-function.

Moreover, the "symmetric and antisymmetric Laplace-type transforms"

$$\tilde{H}^{(s)} = \tilde{H}_s + \tilde{H}_u \quad , \quad \tilde{H}^{(a)} = \tilde{H}_s - \tilde{H}_u, \tag{3.53}$$

which are defined in terms of $\Delta^{(s)}\underline{H}_{(\zeta,\zeta',t)}$ and $\Delta^{(a)}\underline{H}_{(\zeta,\zeta',t)}$ (see Eqs (3.45), (3.46)) via the similar formulae:

$$\tilde{H}^{(s),(a)}\left(\zeta,\zeta';t,\lambda_t\right) = \omega_{d-1}\int_0^{\infty} \Delta^{(s),(a)} \underline{H}_{(\zeta,\zeta',t)}\left(\cosh v\right) Q^{(d)}_{\lambda_t}\left(\cosh v\right)\left(\sinh v\right)^{d-2} dv \tag{3.54}$$

enjoy the following "*Froissart–Gribov-type equalities*"

$$\text{for } \ 2\ell > N_H \qquad \tilde{H}^{(s)}\left(\zeta,\zeta',t,2\ell\right) = \tilde{h}_{2\ell}\left(\zeta,\zeta',t\right) \tag{3.55}$$

$$\text{for } \ 2\ell+1 > N_H, \qquad \tilde{H}^{(a)}\left(\zeta,\zeta',t,2\ell+1\right) = \tilde{h}_{2\ell+1}\left(\zeta,\zeta',t\right) \tag{3.56}$$

$\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ are *Carlsonian* [5] (i.e. unique) *interpolations* of the respective sets of partial waves $\left\{\tilde{h}_{2\ell};\ 2\ell > N_H\right\}$ and $\left\{\tilde{h}_{2\ell+1};\ 2\ell+1 > N_H\right\}$; they indeed satisfy bounds of the following form in $\mathbb{C}^{(N_H+\varepsilon)}_{+}$ (for all positive ε,ε') :

$$\left|\tilde{H}^{(s),(a)}\left(\zeta,\zeta';t,\lambda_t\right)\right| \leq c^{(s),(a)}_{(\varepsilon,\varepsilon')}\ C^{(H)}_{(\zeta,\zeta',t)}\ \left|\lambda_t - N_H\right|^{n_H - \frac{d-2}{2}+\varepsilon'}\ \mathrm{e}^{-[\Re e\lambda_t - (N_H+\varepsilon)]v_0} \tag{3.57}$$

where $v_0 = \min\left(v_s\left(\zeta,\zeta',t\right), v_u\left(\zeta,\zeta',t\right)\right)$, v_s, v_u being the quantities introduced in (3.29), (3.30).

These bounds on $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ reexpress exactly (after symmetrization and antisymetrization) the bounds (4.84) established in theorem 5 of [1]; in particular, the occurrence of the power of $|\lambda_t - N_H|$ is correlated to the distribution character of $\Delta^{(s),(a)}\underline{H}_{(\zeta,\zeta',t)}\,(\cosh v)$ encoded in the bound (3.27). However, we now have rewritten the constants $C^{\varepsilon,\varepsilon'}_{s,u}$ of the latter reference under the form $c^{(s),(a)}_{(\varepsilon,\varepsilon')}\ C^{(H)}_{(\zeta,\zeta',t)}$, where the quantity $C^{(H)}_{(\zeta,\zeta',t)}$ (see Eq (3.28)) contains the full dependence of the bound (3.27) with respect to the mass-variables ζ, ζ', while $c^{(s)}_{(\varepsilon,\varepsilon')}$ and $c^{(a)}_{(\varepsilon,\varepsilon')}$ are purely numerical constants: the dependence of the bound (3.57) with respect to $C^{(H)}_{(\zeta,\zeta',t)}$, results from the linearity of the transformations which associate $\tilde{H}^{(s)}(\zeta,\zeta';t,\lambda_t)$ and $\tilde{H}^{(a)}(\zeta,\zeta';t,\lambda_t)$ with $\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$.

Since $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ provide interpolations in the variable λ_t of the respective sequences of *even* and *odd* Euclidean partial waves $\{\tilde{h}_{2\ell}(\zeta,\zeta',t);\ 2\ell > N_H\}$ and $\{\tilde{h}_{2\ell+1}(\zeta,\zeta',t);\ 2\ell+1 > N_H\}$ of $H([k])$, it is now natural to consider the corresponding interpolations in λ_t (for $\Re e\lambda_t > N_H$) of the BS-equations (3.26) for these partial waves, which can be written as follows:

$$\tilde{H}^{(s),(a)}\left(\zeta,\zeta';t,\lambda_t\right) = \tilde{B}^{(s),(a)}\left(\zeta,\zeta';t,\lambda_t\right) +$$

$$\int\limits_{\Delta_t} \tilde{B}^{(s),(a)}\left(\zeta,\zeta'';t,\lambda_t\right) \tilde{H}^{(s),(a)}\left(\zeta'',\zeta';t,\lambda_t\right) \underline{G}\left(\zeta''\right) d\mu_t\left(\zeta''\right). \tag{3.58}$$

We have thus obtained two decoupled Bethe–Salpeter-type equations for $\tilde{B}^{(s)}$ and for $\tilde{B}^{(a)}$ in terms of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ respectively, in which the Fredholm integration space reduces to the two-dimensional real region Δ_t of the plane of squared-mass variables $\zeta'' = \left(\zeta''_1, \zeta''_2\right)$, while (t,λ_t) are parameters varying in $\mathbb{R}^- \times \mathbb{C}^{(N_H)}_+$.

In view of the mass-dependence of the uniform bounds (3.57) on $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$, the Fredholm method can again be applied [4] in the equivalent, but simpler form specified in the Appendix, where the *variable* integration space $\{\zeta \in \Delta_t\}$ is replaced by the *fixed* integration space $\{(\rho'', w'') \in \mathbb{R}^+ \times \mathbb{R}\}$. This proves the existence of the solutions $\tilde{B}^{(s)}(\zeta,\zeta';t,\lambda_t)$ and $\tilde{B}^{(a)}(\zeta,\zeta';t,\lambda_t)$ of the integral equations (3.58), depending meromorphically on the parameters t and λ_t for (t,λ_t) varying in $\mathbb{R}^- \times \mathbb{C}^{(N_H)}_+$.

By construction, these functions $\tilde{B}^{(s)}$ and $\tilde{B}^{(a)}$ are interpolations in the λ_t−plane of the corresponding Euclidean partial waves $\{\tilde{b}_{2\ell}(\zeta,\zeta',t);\ 2\ell > N_H\}$ and $\{\tilde{b}_{2\ell+1}(\zeta,\zeta',t);\ 2\ell+1 > N_H\}$ of the Bethe-Salpeter kernel $B([k])$, namely there holds the following Froissart-Gribov-type equalities:

$$\text{for}\ \ 2\ell > N_H, \quad \tilde{B}^{(s)}(\zeta,\zeta';t,2\ell) = \tilde{b}_{2\ell}(\zeta,\zeta',t), \tag{3.59}$$

[4]Note that in the bound (3.57) the mass dependence has to be majorized by $C^{(H)}_{(\zeta,\zeta',t)}$ since the exponential factor (also mass-dependent through v_0) has no better uniform majorant than 1. In the treatment of the Appendix one then uses a majorant of $C^{(H)}_{(\zeta,\zeta',t)}$ of the form (A.1) as explained there.

$$\text{for } \; 2\ell+1 > N_H, \quad \tilde{B}^{(a)}(\zeta,\zeta';t,2\ell+1) = \tilde{b}_{2\ell+1}(\zeta,\zeta',t), \tag{3.60}$$

According to the results of Proposition A3, in which F is replaced by $\tilde{H}^{(s)}$ (resp, $\tilde{H}^{(a)}$), we can make the following remark on the corresponding resolvent $R_{F|\alpha=-1} = \tilde{B}^{(s)}$ (resp. $\tilde{B}^{(a)}$).

Remark the general bounds (3.57) on $\tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t)$ lead one to a choice of the function $C(\lambda_t)$ of Proposition A3 proportional to $|\lambda_t - N_H|^{n_H - \frac{d-2}{2}}$, which implies bounds of the type (A.22) on $\tilde{B}^{(s),(a)}(\zeta,\zeta';t,\lambda_t) \times \mathcal{D}_{\tilde{H}^{(s)(a)}|\alpha=-1}(t,\lambda_t)$. These bounds are comparable to (3.57) concerning their dependence in the mass variables, but contain in general no temperate dependence with respect to λ_t in $\mathbb{C}_+^{(N_H)}$; this also corresponds to the impossibility of controlling a priori the poles of $\tilde{B}^{(s),(a)}$ in the limit $|\lambda_t| \to \infty$.

In the following, we shall always be led to assume that B satisfies an extra-assumption of temperateness which leads us to the following statement

Proposition 1 *If $\Delta^{(s)}\underline{B}_{(\zeta,\zeta',t)}(\cosh v)$ and $\Delta^{(a)}\underline{B}_{(\zeta,\zeta',t)}(\cosh v)$ have a behaviour at infinity which is governed by $\mathrm{e}^{N_B v}$ (with $N_B \geq N_H$), then the solutions $\tilde{B}^{(s)}$ and $\tilde{B}^{(a)}$ of the BS-equation (3.58) are holomorphic functions of λ_t in the half-plane $\mathbb{C}_+^{(N_B)}$, and coincide there with the following L_d-transforms*

$$\underline{\tilde{B}}^{(s),(a)}(\zeta,\zeta';t,\lambda_t) = \omega_{d-1}\int_0^\infty \Delta^{(s)(a)}\underline{B}_{(\zeta,\zeta',t)}(\cosh v) Q_{\lambda_t}^{(d)}(\cosh v)(\sinh v)^{d-2}dv \tag{3.61}$$

Moreover, the BS-equations (3.58) then appear as "diagonalized forms" of the original BS-equations (3.47),(3.48).

For simplicity, we give the proof of this result for the case when $\Delta^{(s),(a)}\underline{B}_{(\zeta,\zeta',t)}$ are locally integrable functions. Applying the L_d-transformation (3.50) side-by-side to Eqs (3.47),(3.48), one obtains for λ_t in $\mathbb{C}_+^{(N_B)}$ (in view of Eqs (3.54), (3.61)):

$$\tilde{H}^{(s),(a)}\left(\zeta,\zeta';t,\lambda_t\right) = \underline{\tilde{B}}^{(s),(a)}\left(\zeta,\zeta';t,\lambda_t\right) + \omega_{d-1}\int_0^\infty Q_{\lambda_t}^{(d)}(\cosh v)(\sinh v)^{d-2}\times\ldots$$

$$\left[\int_{\Delta_t}\left(\Delta^{(s),(a)}\underline{B}_{(\zeta,\zeta'',t)} \blacklozenge \Delta^{(s),(a)}\underline{H}_{(\zeta'',\zeta',t)}\right)(\cosh v) \quad \underline{G}\left(\zeta''\right)d\mu_t\left(\zeta''\right)\right]dv \tag{3.62}$$

We shall apply the result of [12] (proposition III-2) according to which the L_d-transform of the convolution product $F_1 \lozenge F_2$ of invariant Volterra kernels $F_i\left(z,z'\right) = f_i\left(-z\cdot z'\right), i=1,2$, namely $f_1 \blacklozenge f_2$ (in view of our definition (3.35)), is equal to the product $\tilde{F}_1(\lambda)\times\tilde{F}_2(\lambda)$.

More precisely (see proposition II-2i) of [12]), if the kernels F_i $(i = 1, 2)$ are regular functions satisfying norm conditions of the form

$$g_N(f_i) = \int_0^\infty \mathrm{e}^{-Nv} \; |f_i(\cosh v)| dv \; < \; \infty, \tag{3.63}$$

then their Volterra convolution product satisfies a bound of the form

$$g_N(f_1 \blacklozenge f_2) \leq c_N \; g_N(f_1) \; g_N(f_2) \; < \; \infty, \tag{3.64}$$

where c_N is a numerical constant. Correspondingly, the product $\tilde{F}_1 \times \tilde{F}_2$ is holomorphic in $\mathbb{C}_+^{(N)}$ and one has:

$$|\tilde{F}_1(\lambda)\tilde{F}_2(\lambda)| \leq \mathrm{cst} \int_0^\infty |(f_1 \blacklozenge f_2)(\cosh v)| \; |Q_\lambda^{(d)}(\cosh v)| (\sinh v)^{d-2} dv$$

$$\leq \; c_{(\varepsilon)} \; g_N(f_1 \blacklozenge f_2), \tag{3.65}$$

uniformly in any half-plane $\mathbb{C}_+^{(N+\varepsilon)}$ (for all $\varepsilon > 0$ and a suitable choice of the constant $c_{(\varepsilon)}$): this majorization follows from the exponential decrease property of the function $Q_\lambda^{(d)}(\cosh v)$ (see [12]).

Assuming that the absorptive parts $\Delta^{(s),(a)}H$, $\Delta^{(s),(a)}B$ satisfy norm conditions of the form (3.63) with the mass dependence given by Eq. (3.28), namely $g_{N_B}(\Delta^{(s),(a)}\underline{H}_{(\zeta,\zeta',t)}) = c_H \; C^{(H)}_{(\zeta,\zeta',t)}$ (as it is implied by (3.27) if $n_H = 0$) and $g_{N_B}(\Delta^{(s),(a)}\underline{B}_{(\zeta,\zeta',t)}) = c_B \; C_{(\zeta,\zeta',t)}$, ($c_H$ and c_B being numerical constants), then it follows from (3.64),(3.65) that the repeated integral

$$\int\limits_{\Delta_t} \left[\int_0^\infty \left(\Delta^{(s),(a)}\underline{B}_{(\zeta,\zeta'',t)} \; \blacklozenge \; \Delta^{(s),(a)}\underline{H}_{(\zeta'',\zeta',t)} \right) (\cosh v) \, Q_{\lambda_t}^{(d)}(\cosh v)(\sinh v)^{d-2} dv \right]$$

$$\ldots \times \underline{G}\left(\zeta''\right) d\mu_t\left(\zeta''\right) \tag{3.66}$$

is absolutely convergent and therefore equal to the double integral at the r.h.s. of Eq.(3.62). The integral over v in (3.66) is the L_d-transform of a Volterra convolution, equal to the product $\underline{\tilde{B}}^{(s),(a)}\left(\zeta, \zeta''; t, \lambda_t\right)$ $\tilde{H}^{(s),(a)}\left(\zeta'', \zeta'; t, \lambda_t\right)$ in $\mathbb{C}_+^{(N_B)}$. This shows that Eq.(3.62) coincides with the corresponding BS-equation (3.58) for the Laplace transforms, so that $\underline{\tilde{B}}^{(s),(a)}$ coincides with the restriction of $\tilde{B}^{(s),(a)}$ to the half-plane $\mathbb{C}_+^{(N_B)}$.

The case when $\Delta^{(s),(a)}\underline{B}_{(\zeta,\zeta'',t)}$ are distributions can be treated similarly, since the convolution product $F_1 \lozenge F_2$ of distribution-like invariant Volterra kernels is again transformed by L_d in the corresponding product $\tilde{F}_1(\lambda) \times \tilde{F}_2(\lambda)$, these holomorphic functions admitting now a power bound in $|\lambda|$ instead of being bounded.

4 On the Bethe-Salpeter generation of Regge poles in general quantum field theory

In Sec 3 we have exhibited how the BS-type structure resulting from the general (axiomatic) framework of QFT can be expressed in terms of the squared-mass and angular variables, and then in terms of the squared-mass and *complex angular momentum* variables; at each step, this was done by considering the kernel B (resp. $\tilde{B}^{(s),(a)}$) as determined by H via the Fredholm method. In the present section, we shall adopt a more exploratory viewpoint by assuming that in the field theory under consideration, the kernel $B([k]) = B[t;\rho,w,\rho',w';\cos\Theta_t]$ satisfies "better properties" than H, in a sense to be specified below. We then intend to draw the consequences of these additional assumptions on B for the structure of H.

4.1 Local generation of Regge poles

We consider again the diagonalized form (3.58) of the Bethe-Salpeter structure of a given four-point function H, assuming that the conditions of Proposition 1 are satisfied with $N_B = N_H$. Eq. (3.58) is then valid as an identity between holomorphic functions of λ_t in the half-plane $\mathbb{C}_+^{(N_H)}$, for all values of (ζ,ζ') in $\Delta_t \times \Delta_t$ and negative t. Moreover $\tilde{B}^{(s),(a)}$ satisfy bounds which are similar to those on $\tilde{H}^{(s),(a)}$, as far as their dependence on the mass variables ζ,ζ' is concerned, namely these bounds contain (as in (3.57)) a factor $C_{(\zeta,\zeta',t)}$ of the form (3.28).

Let us now make the additional assumption that $\tilde{B}^{(s),(a)}$ can be analytically continued in some disk of the λ_t−plane, centered on the border line ($\Re e\lambda_t = N_H$) of $\mathbb{C}_+^{(N_H)}$ and still satisfy similar bounds including the factor $C_{(\zeta,\zeta',t)}$ in that disk. Let D be the intersection of the latter with the closed left-hand plane $\Re e\lambda_t \le N_H$; we can consider the analytic continuation of (3.58) in D, these equations being now regarded as Fredholm-resolvent equations defining $\tilde{H}^{(s)}$ (resp. $\tilde{H}^{(a)}$) in terms of $\tilde{B}^{(s)}$ (resp. $\tilde{B}^{(a)}$) through expressions of the form:

$$\tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t) = \frac{\mathcal{N}_{\tilde{B}^{(s),(a)}}(\zeta,\zeta';t,\lambda_t)}{\mathcal{D}_{\tilde{B}^{(s),(a)}}(t,\lambda_t)}. \tag{4.1}$$

In Eq. (4.1), the notations of Proposition A3 have been simplified by putting

$$\mathcal{N}_{\tilde{B}^{(s),(a)}}(\zeta,\zeta';t,\lambda_t) = \mathcal{N}_{\tilde{B}^{(s),(a)}|\alpha=1}(t,\lambda_t;\rho,w,\rho',w') \quad \text{and}$$

$$\mathcal{D}_{\tilde{B}^{(s),(a)}}(t,\lambda_t) = \mathcal{D}_{\tilde{B}^{(s),(a)}|\alpha=1}(t,\lambda_t).$$

Since $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ are holomorphic in $\mathbb{C}_+^{(N_H)}$, the resolution (4.1) provides a meromorphic continuation of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ in D, whose poles are given respectively by the zeros of $\mathcal{D}_{\tilde{B}^{(s)}(t,\lambda_t)}$ and $\mathcal{D}_{\tilde{B}^{(a)}(t,\lambda_t)}$: the locations of the latter in the λ_t−plane will be denoted respectively by $\lambda_t = \lambda_{j^{(s)}}(t), \lambda_t = \lambda_{j^{(a)}}(t)$.

Remark The variable t being kept real (and negative) at the present step of our program, there is no point of proving the analytic dependence of the previous zeros with respect to t; however, this analyticity will naturally appear at a further step, once one has extended the results of [1] to a relevant set of complex values of t by techniques of analytic completion.

For simplicity, we shall only consider the case when there is a unique zero $\lambda^{(s)}(t)$ (resp. $\lambda^{(a)}(t)$) of $\mathcal{D}_{\tilde{B}^{(s)}(t,\lambda_t)}$ (resp. $\mathcal{D}_{\tilde{B}^{(a)}(t,\lambda_t)}$): in D, which we suppose to be a *simple* zero; correspondingly, $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ admit these values $\lambda^{(s)}(t)$, $\lambda^{(a)}(t)$ as simple poles, which can be called *Regge poles*, and we introduce the corresponding residue functions:

$$\mathrm{Res}\ \tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda^{(s),(a)}(t))$$

$$= [(\lambda_t - \lambda^{(s),(a)}(t)) \times \tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t)]_{|\lambda_t=\lambda^{(s),(a)}(t)}. \tag{4.2}$$

Factorization property of the Regge pole residues:

We claim that, as a result of Fredholm theory, the functions Res $\tilde{H}^{(s)}$ and Res $\tilde{H}^{(a)}$ can be written under the following form:

$$\mathrm{Res}\ \tilde{H}^{(s)}(\zeta,\zeta';t,\lambda^{(s)}(t)) = \frac{p^{(s)}(\zeta,\zeta',t)}{\beta^{(s)}(t)}, \tag{4.3}$$

$$\mathrm{Res}\ \tilde{H}^{(a)}(\zeta,\zeta';t,\lambda^{(a)}(t)) = \frac{p^{(a)}(\zeta,\zeta',t)}{\beta^{(a)}(t)}, \tag{4.4}$$

where $p^{(s)}$ and $p^{(a)}$ are (for each t) *kernels of finite rank* satisfying the *projector relations* $p^{(s)} \circ_t p^{(s)} = p^{(s)}$, $p^{(a)} \circ_t p^{(a)} = p^{(a)}$. This can be seen as follows.

Making use of the expression (A.16) of the Fredholm resolvent $R_{\tilde{B}^{(s)}}$ of $\tilde{B}^{(s)}$, and calling $\alpha = \underline{\alpha}(t,\lambda_t)$ the zero of $\mathcal{D}_{\tilde{B}^{(s)}}(t,\lambda_t;\alpha)$ whose restriction to $\alpha = 1$ yields the pole $\lambda_t = \lambda^{(s)}(t)$ of $\tilde{H}^{(s)}$ (λ_t being thus a solution of the implicit equation $\underline{\alpha}(t,\lambda_t) - 1 = 0$), one can show (see e.g. Sec 3.3 of [17] and references therein to standard results of Fredholm theory) that

$$R_{\tilde{B}^{(s)}}(\zeta,\zeta';t,\lambda_t,\alpha) = \frac{p^{(s)}(\zeta,\zeta',t)}{\alpha - \underline{\alpha}(t,\lambda_t)} + R'_{\tilde{B}^{(s)}}(\zeta,\zeta';t,\lambda_t,\alpha),$$

where $p^{(s)}$ is of finite rank and $R'_{\tilde{B}^{(s)}}$ holomorphic in α at $\alpha = \underline{\alpha}(t,\lambda_t)$. Formula (4.3) then follows by putting $\beta^{(s)}(t) = -\frac{\partial\underline{\alpha}}{\partial\lambda_t}(t,\lambda^{(s)}(t))$. The analysis would be more involved in the case of multiple poles (not considered here).

By assumption, we shall consider as "generic" the case when the Regge pole terms are characterized by *projectors of rank one*; for the case of Hermitian fields ϕ_1, ϕ_2 considered here, these terms can always be written under the following form:

$$p^{(s)}(\zeta,\zeta',t) = \overline{\varphi}^{(s)}(\zeta,t) \times \varphi^{(s)}(\zeta',t), \tag{4.5}$$

$$p^{(a)}(\zeta,\zeta',t) = \overline{\varphi}^{(a)}(\zeta,t) \times \varphi^{(a)}(\zeta',t), \tag{4.6}$$

with $\int_{\Delta_t} |\varphi^{(s)}(\zeta,t)|^2 d\mu_t(\zeta) = \int_{\Delta_t} |\varphi^{(a)}(\zeta,t)|^2 d\mu_t(\zeta) = 1$. The Hermitian character of the r.h.s. of Eqs (4.5), (4.6) is in fact implied by the symmetry property $\tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t) = \overline{\tilde{H}^{(s),(a)}(\zeta,\zeta';t,\overline{\lambda}_t)}$ together with the assumption that $\lambda^{(s),(a)}(t)$ are real. To be more precise, if $H([k])$ is the four-point function of two *Hermitian* scalar fields ϕ_1, ϕ_2, it satisfies in its analyticity domain the symmetry relation $H(K;Z,Z') = \overline{H(\overline{K};\overline{Z}',\overline{Z})}$. The latter implies the following ones (for real values of t,ζ,ζ'): $\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t) = \overline{\underline{H}_{(\zeta,\zeta',t)}(\overline{\cos\Theta_t})}$ and therefore also (in view of Eq. (3.54)): $\tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t) = \overline{\tilde{H}^{(s),(a)}(\zeta,\zeta';t,\overline{\lambda_t})}$. It follows that the poles $\lambda_t = \lambda_{j^{(s),(a)}}(t)$ resulting from Eq. (4.1) can only be produced either at real values or at pairs of complex conjugate values. However in the range $t < 0$ considered here, the case of *real* poles appears to be more "physical", as it can be illustrated by simple models of BS-kernels, and this justifies our reality assumption on $\lambda^{(s),(a)}(t)$.

Remark The functions $\varphi^{(s),(a)}(\zeta,t)$ which depend on the three Lorentz invariants ζ_1,ζ_2,t can be interpreted (from the geometrical viewpoint) as *three-point functions* defined in the Euclidean domain of field theory, namely in the set of configurations $(k_1,k_2,k_3 = k_1+k_2)$ such that $k_i = (iq_i^{(0)},\vec{p}_i),\ i=1,2,3$, whose invariants $\zeta_1 = k_1^2,\ \zeta_2 = k_2^2$ and $t = (k_1+k_2)^2$ vary in the set $\{(\zeta,t);\ \zeta \in \Delta(t),\ t<0\}$.

4.2 Asymptotic assumption on the Bethe-Salpeter kernel and generation of dominant Reggeon terms in the four-point functions:

In this subsection, we shall assume that $B([k]) = B[t;\rho,w,\rho',w';\cos\Theta_t]$ satisfies increase properties in the variable $\cos\Theta_t = -z\cdot z'$ which are *"better than"* those of H. We then intend to show that, for any value of t for which this assumption is made, there exists a possible Regge pole structure of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ which will produce corresponding asymptotically dominant "Reggeon terms" in the four-point function $H([k])$. For the negative values of the channel variable t only considered in the present paper, it appears difficult to give a general field-theoretical support to this type of assumption, although it is naturally satisfied by standard approximations of the Bethe-Salpeter kernel B such as the relativistic counterpart of Yukawa-type potentials. As a matter of fact, it is for t varying in some *positive* interval that we shall be able to produce more general arguments in favour of such assumption on B, but this can only be relevant in our program after the analytic continuation in the variable t has been performed (in a further work). However, since the mathematical analysis is independent of the value of t (playing the role of a fixed parameter), we think it appropriate to present it below as *"a generic procedure for the generation of asymptotically dominant Reggeon terms in the field-theoretical framework"*.

More specifically, we shall now assume that B satisfies the following bounds involving a given exponent N_B and a positive constant $C_E^{(B)}$:

i) for all u (($0 \le u \le \pi$ and $\pi \le u \le 2\pi$, the jumps at $u = 0$ and $u = \pi$ being taken into account),

$$\int_0^\infty \mathrm{e}^{-N_B v} |\underline{B}_{(\zeta,\zeta',t)}(\cos(u+iv))| dv \le C^{(B)}_{(\zeta,\zeta',t)}, \tag{4.7}$$

$$ii) \qquad \sup_{-1 \le \cos\Theta_t \le 1} |\underline{B}_{(\zeta,\zeta',t)}(\cos\Theta_t)| \le C^{(B)}_{(\zeta,\zeta',t)}, \tag{4.8}$$

with

$$C^{(B)}_{(\zeta,\zeta',t)} = C_E^{(B)} [(1+|t|^{\frac{1}{2}})\ (1+\rho)\ (1+|w||t|^{\frac{1}{2}})\ (1+\rho')\ (1+|w'||t|^{\frac{1}{2}})]^{N_B}. \tag{4.9}$$

In these assumptions, *the real number N_B is supposed to satisfy the conditions* $\max(-1, -\frac{d-2}{2}) < N_B < N_H$. So, if we compare the previous assumptions on B with the corresponding bounds (3.27), (3.28) on H, we see that they differ under two respects:

a) the increase at infinity in the $\cos\Theta_t-$plane is governed by $\mathrm{e}^{N_B|\Im m\Theta_t|}$ (dominated by $\mathrm{e}^{N_H|\Im m\Theta_t|}$),

b) the use of an L^1-bound instead of a uniform bound (as for H in (3.27)) is motivated by its convenience for the $\star^{(c)}-$convolution formalism on $X_d^{(c)}$. Here, the local order of singularity of the boundary values and discontinuities $\Delta^{(s),(a)}B$ of B is encoded in their L^1-character with respect to the variable $v = \Im m\Theta_t$ together with their uniform dependence on $\sin\Re e\Theta_t$.

We then have:

Proposition 2 *The previous assumptions i), ii) on B imply the analyticity property with respect to λ_t in $\mathbb{C}_+^{(N_B)}$ and the following majorizations for the transforms $\tilde{B}^{(s)}$, $\tilde{B}^{(a)}$ of $\Delta^{(s)}B$, $\Delta^{(a)}B$:*

$$|\tilde{B}^{(s),(a)}(\zeta,\zeta';t,\lambda_t)| \le C^{(B)}_{(\zeta,\zeta',t)} \Psi(|\Im m\lambda_t|), \tag{4.10}$$

where Ψ denotes a bounded positive function on $[0,\infty[$, tending to zero at infinity. These majorizations hold uniformly for all $(\zeta,\zeta') \in \Delta_t \times \Delta_t$, $t < 0$ and $\lambda_t \in \mathbb{C}_+^{(N_B)}$.

The proof of the latter relies on the fact that the action of the L_d-transformation on $\Delta^{(s),(a)}B$ factorizes as follows (see [12], formulae (III-1),(III-3)):

$$\tilde{B}^{(s),(a)}(\zeta,\zeta';t,\lambda_t) = \int_{\underline{v}}^\infty \mathrm{e}^{-\lambda_t w} \widehat{\Delta^{(s),(a)}} B(\zeta,\zeta';t,w) dw, \tag{4.11}$$

with

$$\Delta^{\widehat{(s),(a)}} B(\zeta,\zeta';t,w) = \cdots$$

$$\omega_{d-2} e^{-\frac{d-2}{2}} \int_0^w \Delta^{(s),(a)} \underline{B}_{(\zeta,\zeta',t)}(\cosh v)[2(\cosh w - \cosh v)]^{\frac{d-4}{2}} \sinh v dv. \quad (4.12)$$

It can be seen (see corollary of Proposition II-6 in [12]) that in view of the condition $N_B > -1$ the L^1-bound on $\Delta^{(s),(a)}\underline{B}_{(\zeta,\zeta',t)}(\cosh v)$ deduced from (4.7), namely

$$\int_0^\infty e^{-N_B v} \ |\Delta^{(s),(a)}\underline{B}_{(\zeta,\zeta',t)}(\cosh v)| \ dv \le C^{(B)}_{(\zeta,\zeta',t)} \quad (4.13)$$

implies the same L^1-bound (up to a constant factor) on $\Delta^{\widehat{(s),(a)}} B(\zeta,\zeta';t,w)$ Then since Eq. (4.11) represents a usual Fourier-Laplace transformation, the announced bound (4.10) readily follows from the Lebesgue theorem.

Remark a similar result holds if the bound (4.13) is satisfied only in the sense of distributions: this is the case e.g. if B contains a simple or multiple pole of the form $(\cos\Theta_t - \cosh v_0)^{-q}$, interpreted as a "particle-exhange" (or Yukawa-type) contribution for $q = 1$, or as a more singular "gluon-type exchange" contribution for $q > 1$.

in view of Proposition 2, the considerations of subsection 4.1 apply to the analytic continuations of the kernels $\tilde{B}^{(s)}$ and $\tilde{B}^{(a)}$ in the full half-plane $\Re e\lambda_t > N_B$. It follows that $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ admit meromorphic continuations of the form (4.1) in the strip $N_B < \Re e\lambda_t \le N_H$. In this strip, there will occur possible Regge poles equipped with factorized residues of the form described by Eqs (4.3),..,(4.6). Moreover we notice that, in view of the bounds (4.10) on $\tilde{B}^{(s),(a)}$, the last statement of Proposition A3 can be applied. It follows that, under our assumptions on B, the poles $\lambda_t = \lambda_{j^{(s)}}(t), \lambda_t = \lambda_{j^{(a)}}(t)$ of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ are confined in a bounded region of the form $\{\lambda_t;\ N_B < \Re e\lambda_t \le N_H,\ |\Im m\lambda_t| < \nu_\Psi(t)\}$. Finally, by applying the majorization (A.22) with $C(\lambda_t) = C_E^{(B)}\Psi(|\Im m\lambda_t|)$, we obtain bounds of the following form which hold uniformly for all $(\zeta,\zeta') \in \Delta_t \times \Delta_t$, $N_B < \Re e\lambda_t \le N_H$, (and $t \le -\varepsilon$, for any given ε, $\varepsilon > 0$):

$$|\mathcal{D}_{B^{(s),(a)}}(t,\lambda_t) \times \tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t)| \le \hat{C}^{(B)}_{(\zeta,\zeta',t)} \ \Psi(|\Im m\lambda_t|), \quad (4.14)$$

where $\hat{C}^{(B)}_{(\zeta,\zeta',t)}$ is given by Eq. (A.22). Besides, for $|\Im m\lambda_t| > A(t)$, with $A(t)$ sufficiently large, and $N_B < \Re e\lambda_t \le N_H$, one can satisfy the inequality $|\mathcal{D}_{B^{(s),(a)}}(t,\lambda_t) - 1| < \frac{1}{2}$ (in view of (A.14)), and therefore the bound (4.14) can be advantageously replaced by

$$|\tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t)| \le 2\hat{C}^{(B)}_{(\zeta,\zeta',t)}\Psi(|\Im m\lambda_t|). \quad (4.15)$$

Reggeon structure of the four-point function:

We shall now show that the previous properties of meromorphic continuation of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ imply the existence of a peculiar structure of the four-point function $H([k]) \equiv \underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$ as a function of t and $\cos\Theta_t$. This will be done by starting from the inversion formula (see Eq (4.89) in Theorem 5 of [1]), which allows one to reexpress $\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$ *for all* $\cos\Theta_t$ *in the cut-plane* $\Pi = \mathbb{C} \setminus \{[\cosh v_0, +\infty[\ \cup\ ]-\infty, -\cosh v_0]\}$ in terms of $\tilde{H}_s = \frac{\tilde{H}^{(s)}+\tilde{H}^{(a)}}{2}$ and $\tilde{H}_u = \frac{\tilde{H}^{(s)}-\tilde{H}^{(a)}}{2}$. In view of the bounds (3.27) on H and correspondingly of the bounds (3.57) on $\tilde{H}^{(s)}$,$\tilde{H}^{(a)}$ in $\mathbb{C}_+^{(N_H)}$, formula (4.89) of [1] can be applied [5] and yields:

$$\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t) =$$

$$-\frac{1}{2i\omega_d}\int_{N_H+\varepsilon-i\infty}^{N_H+\varepsilon+i\infty} \tilde{H}^{(s)}(\zeta,\zeta';t,\lambda)\, \frac{\mathrm{h}_d(\lambda)[P_\lambda^{(d)}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)+P_\lambda^{(d)}(\cos(\Theta_t)]}{2\sin\pi\lambda}d\lambda$$

$$-\frac{1}{2i\omega_d}\int_{N_H+\varepsilon-i\infty}^{N_H+\varepsilon+i\infty} \tilde{H}^{(a)}(\zeta,\zeta';t,\lambda)\, \frac{\mathrm{h}_d(\lambda)[P_\lambda^{(d)}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)-P_\lambda^{(d)}(\cos(\Theta_t)]}{2\sin\pi\lambda}d\lambda$$

$$+\frac{1}{\omega_d}\sum_{0\le\ell\le N_H} \tilde{h}_\ell(\zeta,\zeta',t)\ \mathrm{h}_d(\ell)P_\ell^{(d)}(\cos\Theta_t). \tag{4.16}$$

In the latter, $\mathrm{h}_d(\lambda) = \frac{2\lambda+d-2}{(d-2)!}\times\frac{\Gamma(\lambda+d-2)}{\Gamma(\lambda+1)}$, $\varepsilon(\Theta_t) = \mathrm{sgn}(\Re e\Theta_t)$ and $P_\lambda^{(d)}(\cos\Theta_t)$ denotes the first-kind Legendre function in dimension d , whose integral representation is given by Eq. (4.64) of [1].

The absolute convergence of the integral at the r.h.s. of Eq. (4.16) is ensured for all $\cos\Theta_t$ in Π (uniformly in $\cos\Theta_t$) by appropriate exponential decrease properties of the integration kernels with respect to the variable $\nu = \Im m\lambda$. These properties result from the following bounds on the Legendre functions $P_\lambda^{(d)}$ (obtained e.g. from formulae (II.85), (II.86) of [12] by a refinement of the argument of Proposition II-12):

$$\left|\frac{\mathrm{h}_d(N+i\nu)P_{N+i\nu}^{(d)}(\cos\Theta_t)}{\sin\pi(N+i\nu)}\right| \le C_d(\cos\Theta_t)\mathrm{e}^{\max(N,-N-d+2)|\Im m\Theta_t|}\mathrm{e}^{-|\nu|(\pi-|\Re e\Theta_t|)}, \tag{4.17}$$

where the function $C_d(\cos\Theta_t)$ is uniformly bounded at infinity in the cut-plane $\mathbb{C}\setminus]-\infty,-1]$.

By now taking advantage of the meromorphic continuation of $\tilde{H}^{(s)}$ and $\tilde{H}^{(a)}$ in the strip $N_B < \Re e\lambda_t \le N_H$ together with the uniform bound (4.15) on these functions when $|\Im m\lambda_t|$ tends to infinity, one can shift the integration line in formula (4.16) from its initial position to the line $\Re e\lambda_t = N_B$, provided one

[5] The choice of the line $\Re e\lambda = N_H + \varepsilon$ (ε arbitrarily small) for the integration cycle in Eq. (4.15) is more correct than the prescription ($\Re e\lambda = N_H$) given in Eq. (4.89) of [1], which necessitates uniform bounds in the closure of $\mathbb{C}_+^{(N_H)}$

extracts residue terms corresponding to all the poles $\lambda_t = \lambda_{j^{(s)}}(t), \lambda_t = \lambda_{j^{(a)}}(t)$ contained in the strip. Assuming there exists only *one simple pole* at $\lambda_t = \lambda^{(s)}(t)$ real for $\tilde{H}^{(s)}$ and *one simple pole* at $\lambda_t = \lambda^{(a)}(t)$ real for $\tilde{H}^{(a)}$, the expression (4.16) of $\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$ can be replaced by:

$$\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t) =$$

$$-\frac{\pi}{\omega_d}\mathrm{Res}\,\tilde{H}^{(s)}(\zeta,\zeta';t,\lambda^{(s)}(t))\frac{\mathrm{h}_d(\lambda^{(s)}(t))\ [P^{(d)}_{\lambda^{(s)}(t)}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)+P^{(d)}_{\lambda^{(s)}(t)}(\cos(\Theta_t)]}{2\sin\pi\lambda^{(s)}(t)}$$

$$-\frac{\pi}{\omega_d}\mathrm{Res}\,\tilde{H}^{(a)}(\zeta,\zeta';t,\lambda^{(a)}(t))\frac{\mathrm{h}_d(\lambda^{(a)}(t))\ [P^{(d)}_{\lambda^{(a)}(t)}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)-P^{(d)}_{\lambda^{(a)}(t)}(\cos(\Theta_t)]}{2\sin\pi\lambda^{(a)}(t)}$$

$$-\frac{1}{2i\omega_d}\int_{N_B-i\infty}^{N_B+i\infty}\tilde{H}^{(s)}(\zeta,\zeta';t,\lambda)\ \frac{\mathrm{h}_d(\lambda)[P^{(d)}_{\lambda}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)+P^{(d)}_{\lambda}(\cos(\Theta_t)]}{2\sin\pi\lambda}d\lambda$$

$$-\frac{1}{2i\omega_d}\int_{N_B-i\infty}^{N_B+i\infty}\tilde{H}^{(a)}(\zeta,\zeta';t,\lambda)\ \frac{\mathrm{h}_d(\lambda)[P^{(d)}_{\lambda}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)-P^{(d)}_{\lambda}(\cos(\Theta_t)]}{2\sin\pi\lambda}d\lambda$$

$$+\frac{1}{\omega_d}\sum_{0\le\ell\le N_B}\tilde{h}_\ell(\zeta,\zeta',t)\ \mathrm{h}_d(\ell)P^{(d)}_\ell(\cos\Theta_t). \tag{4.18}$$

In the latter, the functions $\mathrm{Res}\,\tilde{H}^{(s),(a)}$ are of the form given by Eqs (4.2),...,(4.6). The important feature of Eq. (4.18) concerns the asymptotic behaviour in the $\cos\Theta_t-$plane of the various terms at its right-hand side. In view of the dependence on $\Im m\Theta_t$ of the bound (4.17), it follows that the two integrals as well as the last term at the r.h.s. of Eq. (4.18) are functions of $\cos\Theta_t$ which are bounded at infinity by $|\cos\Theta_t|^{N_B}$. The first two terms at the r.h.s. of Eq. (4.18) then appear as the leading terms giving the asymptotic behaviour at infinity of $\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$. This asymptotic behaviour exhibits rates of increase which are those of $P^{(d)}_{\lambda^{(s)}}(\pm\cos\Theta_t)$ and $P^{(d)}_{\lambda^{(a)}}(\pm\cos\Theta_t)$, namely respectively $|\cos\theta_t|^{\lambda^{(s)}(t)}$ and $|\cos\theta_t|^{\lambda^{(a)}(t)}$, with $N_B < \lambda^{(s),(a)}(t) \le N_H$. These leading terms in Eq. (4.18) will be called *Reggeon terms*. To summarize, we can state:

Theorem

Let $H([k]) = \underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t)$ *be the analytic four-point function of two (mutually) local Hermitian scalar fields whose Bethe-Salpeter structure in the* $t-$*channel is encoded in a BS-kernel* $B([k]) = \underline{B}_{(\zeta,\zeta',t)}(\cos\Theta_t)$. *Let us assume that for some range of negative values of* t *(or possibly the whole half-line* $t < 0$*),* B *satisfies bounds of the form (4.7),(4.8),(4.9). Then in a generic situation involving only one Reggeon term of each symmetry type,* $\underline{H}_{(\zeta,\zeta',t)}$ *admits a representation of the following form, valid for all* (ζ,ζ') *in* $\Delta_t\times\Delta_t$ *and* $\cos\Theta_t$ *in* Π*:*

$$\underline{H}_{(\zeta,\zeta',t)}(\cos\Theta_t) =$$

$$-\frac{\pi}{\omega_d}\frac{\overline{\varphi}^{(s)}(\zeta,t)\times\varphi^{(s)}(\zeta',t)}{\beta^{(s)}(t)}\frac{\mathrm{h}_d(\lambda^{(s)}(t))\ [P^{(d)}_{\lambda^{(s)}(t)}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)+P^{(d)}_{\lambda^{(s)}(t)}(\cos(\Theta_t)]}{2\sin\pi\lambda^{(s)}(t)}$$

$$-\frac{\pi}{\omega_d}\frac{\overline{\varphi}^{(a)}(\zeta,t)\times\varphi^{(a)}(\zeta',t)}{\beta^{(a)}(t)}\frac{\mathrm{h}_d(\lambda^{(a)}(t))\ [P^{(d)}_{\lambda^{(a)}(t)}(\cos(\Theta_t-\varepsilon_{\Theta_t}\pi)-P^{(d)}_{\lambda^{(a)}(t)}(\cos(\Theta_t)]}{2\sin\pi\lambda^{(a)}(t)}$$

$$+\underline{H}'_{(\zeta,\zeta',t)}(\cos\Theta_t), \tag{4.19}$$

where the residual term $\underline{H}'_{(\zeta,\zeta',t)}(\cos\Theta_t)$ *is bounded at infinity by* $Cst|\cos\Theta_t|^{N_B}$ *in* Π, *while the first two terms exhibit leading behaviours governed respectively by* $|\cos\Theta_t|^{\lambda^{(s)}(t)}$ *and* $|\cos\Theta_t|^{\lambda^{(a)}(t)}$, *with* $\lambda^{(s)}(t) > N_B$ *and* $\lambda^{(a)}(t) > N_B$.

5 Conclusion

In this paper, we have established the following results:

a) the general Bethe-Salpeter structure of four-point functions of scalar fields in a given $t-$channel has been shown to be diagonalized in the corresponding complex angular momentum variable λ_t for all negative values of t and of the squared-mass variables corresponding to Euclidean configurations in complex momentum space . More specifically, Carlsonian interpolations of the Euclidean Bethe-Salpeter equations for even and odd partial waves have been constructed in a half-plane of the form Re $\lambda_t > N$, starting from the corresponding Bethe-Salpeter equations for the s and $u-$channel absorptive parts.

b) any information on B leading to continue analytically these interpolations in some region of the left-hand half-plane Re $\lambda_t \leq N$ results in the "potential generation" of Regge poles, equipped with a factorized residue structure involving Euclidean three-point functions.

c) the existence of a meromorphic continuation of these interpolations in a strip of the form $N_b <$ Re $\lambda_t \leq N$ (with the relevant factorization property of the residues of the poles) has been shown to hold if and only if the Bethe-Salpeter kernel satisfies bounds of the form $|\cos\Theta_t|^{N_B}$ in the complex plane of the off-shell scattering angle Θ_t, which are better than those on the complete four-point function H, namely iff $N_B < N_H$.

Here our analysis has been done for negative values of t by relying on the analyticity domains resulting from the general principles of quantum field theory. It is expected to hold similarly for t in some positive interval $]0, t_0[$, and then to produce the desired interpolation of possible bound states in the $t-$channel, after one has justified the necessary properties of analytic continuation of H (in the spirit of [15]).

In the general arguments of our axiomatic approach, the unspecified number N_H was introduced (see [1]) as a consequence of the temperateness assumption in the field-theoretical framework. In a specific model of field theory, the actual asymptotic behaviour of H in the $\cos\Theta_t-$plane would be governed by a precise function $N_H = N_H(t)$. Then the crucial question of the validity of Reggeon terms with asymptotic dominance in the four-point function H of a general quantum field theory appears to rely basically on the knowledge of the exponent $N_B = N_B(t)$ which governs the asymptotic behaviour of B: such Reggeon terms will appear for all values of t such that $N_B(t) < N_H(t)$.

More questionable is the justification of the inequality $N_B(t) < N_H(t)$, which would require further investigations in the present field-theoretical framework. In the absence of global information on general (regularized or renormalized) versions of the two-particle irreducible kernel B , one can think of exploiting various contributions to B defined in terms of generalized Feynman convolutions enjoying the graph property of "two-particle irreducibility". The simplest ones are of course those associated with "particle-exchange" graphs, which are interpreted as relativistic Yukawa-type potentials; they do satisfy the previous inequality (with $N_B = -1$) and have been widely exploited in the literature, since they directly transfer to particle physics the initial procedure of Regge pole production in potential theory. However, much more complicated contributions to B (of higher-order in the coupling constants) are imposed by the Feynman convolution structure of field theory; typical examples of those have been studied by Mandelstam [18] and others (see e.g. [19] and references therein). The treatment of such contributions which these authors have given by using the methods of (maximally analytic) $S-$matrix theory, has led them to introduce "Regge cuts" in the λ_t-plane, whose effect in some situations seems to wipe out the dominant asymptotic behaviour of the potential-type Regge poles introduced at first. Although no mechanism of production of Regge cuts appears in our general field-theoretical framework, it can be seen in our approach that the contributions considered by the previous authors cannot satisfy the inequality $N_B(t) < N_H(t)$ for negative t, although they hopefully do it for positive t. Of course, such results on partial contributions to B do not allow one to conclude the non-validity of the Bethe-Salpeter generation of Reggeon leading terms at negative t; more global (non-perturbative) information on the field models would be needed for investigating the relevant asymptotic properties of B.

APPENDIX

We give a unified treatment of the various versions of BS-equation presented in Sec.3., in which the integration space either contains or is identical with the space of mass variables $\{\zeta = (\zeta_1, \zeta_2) \in \Delta_t\}$ equivalent to $\{(\rho, w); \rho \geq 0, w \in \mathbb{R}\}$ (see Eqs (3.3)...(3.6)). One is led to use the $\mathcal{N}/\mathcal{D}-$resolution of Fredholm resolvent equations in a complex analysis framework, in the following way.

$F(t, \star; \rho, w, \rho', w', \bullet, \bullet')$ will denote a $t-$dependent kernel acting on the integration space $\mathcal{I} = \{(\rho, w, \bullet); \rho \geq 0, w \in \mathbb{R}, (\bullet \in \Gamma)\}$ $(((\rho, w, \bullet), (\rho', w', \bullet')) \in \mathcal{I} \times \mathcal{I})$. The notation $\bullet$ ($\bullet'$) stands for possible additional variables z (z') which vary on a $(d-1)-$ cycle with compact support Γ of $X^{(c)}_{d-1}$: this is the case for the kernel $H[t; \rho, w, \rho', w'; -z \cdot z'] \equiv \underline{H}_{(\zeta,\zeta',t)}(-z \cdot z')$ (resp. $B[\cdots] \equiv \underline{B}_{(\zeta,\zeta',t)}(\cdots)$) of Sec.3.1 (see Eqs (3.7),(3.8)). Alternatively, the kernel may also depend on the complex parameter λ_t varying in a half-plane $\mathbb{C}^{(N)}_+$, here represented by $\star$:

this is the case for the kernels $\tilde{H}^{(s),(a)}[t,\lambda_t;\rho,w,\rho',w'] \equiv \tilde{H}^{(s),(a)}(\zeta,\zeta';t,\lambda_t)$ (or $\tilde{B}^{(s),(a)}$) (see Eq. (3.54)). Our treatment of the BS-equation is valid for the whole range $t<0$ and allows one to study the regularity of the solution for t tending to $-\infty$ rather than for t tending to zero. Indications for the treatment of the latter are given at the end.

F is assumed to satisfy a bound of the following form:

$$|F(t,\star;\rho,w,\rho',w',\bullet,\bullet')| \leq C_{(\star,\bullet,\bullet')}(1+|t|)^{\hat{N}}(1+\rho)^N(1+|w|)^N(1+\rho')^N(1+|w'|)^N \tag{A.1}$$

and one considers the following Fredholm resolvent equation:

$$R_F(t,\star;\rho,w,\rho',w',\bullet,\bullet';\alpha) = F(t,\star;\rho,w,\rho',w',\bullet,\bullet') + \ i\alpha|t|^{\frac{1}{2}} \times \cdots \tag{A.2}$$

$$\int_{\mathcal{I}} F(t,\star;\rho,w,\rho'',w'',\bullet,\bullet'')R_F(t,\star;\rho'',w'',\rho',w',\bullet'',\bullet';\alpha)G[t;\rho'',w'']{\rho''}^{d-1}d\rho''dw''(d\bullet'');$$

in the latter, the weight $G[t;\rho,w]$ is assumed to satisfy the following uniform bound:

$$|G[t;\rho,w]| \leq c_r^2|t|^{-r}(1+\rho)^{-2r}(1+|w|)^{-2r}, \tag{A.3}$$

with $r > \max\left(N+\frac{d}{2},\hat{N}+\frac{1}{2}\right)$ and $c_r^2<1$.

One checks that the relevant bounds (3.9), (3.27), (3.28) on H and (4.1)...(4.3) on B imply bounds of the form (A.1) with respectively $\hat{N}=\frac{3N}{2}$ for all $t<0$ if $N\geq 0$ and $\hat{N}=\frac{N}{2}$ for all $t<-1$ if $N<0$ (case $N=N_B$). Similarly the bound (3.12) on G implies (A.3) for all $t<0$ (with $c_r=4^rc$).

According to the standard Fredholm argument, one introduces the following entire series in α:

$$\mathcal{N}_F(t,\star;\rho,w,\rho',w',\bullet,\bullet';\alpha) = \sum_{p=0}^{\infty} N^{(p)}(t,\star;\rho,w,\rho',w',\bullet,\bullet')\frac{\alpha^p}{p!}, \tag{A.4}$$

$$\mathcal{D}_F(t,\star;\alpha) = \sum_{p=0}^{\infty} D^{(p)}(t,\star)\frac{\alpha^p}{p!}, \tag{A.5}$$

where $\quad N^{(0)} \ = \ F, \quad D^{(0)} \ = \ 1, \quad$ and for $p\geq 1$:

$$N^{(p)}(t,\star;\rho,w,\rho',w',\bullet,\bullet') = (-i)^p|t|^{\frac{p}{2}} \times \cdots$$

$$\int_{\mathcal{I}^p} \left| \begin{matrix} F(t,\star;\rho,w,\rho',w',\bullet,\bullet') & F(t,\star;\rho,w,\rho_j,w_j,\bullet,\bullet_j) \\ F(t,\star;\rho_i,w_i,\rho',w',\bullet_i,\bullet') & F(t,\star;\rho_i,w_i,\rho_j,w_j,\bullet_i,\bullet_j) \end{matrix} \right|_{1\le i,j\le p} \times \cdots$$

$$G[t;\rho_1,w_1]\cdots G[t;\rho_p,w_p]{\rho_1}^{d-1}d\rho_1 dw_1(d\bullet_1)\cdots {\rho_p}^{d-1}d\rho_p dw_p(d\bullet_p), \qquad \text{(A.6)}$$

$$D^{(p)}(t,\star) = (-i)^p|t|^{\frac{p}{2}} \int_{\mathcal{I}^p} |F(t,\star;\rho_i,w_i,\rho_j,w_j,\bullet_i,\bullet_j)|_{1\le i,j\le p} \times \cdots$$

$$G[t;\rho_1,w_1]\cdots G[t;\rho_p,w_p]{\rho_1}^{d-1}d\rho_1 dw_1(d\bullet_1)\cdots {\rho_p}^{d-1}d\rho_p dw_p(d\bullet_p). \qquad \text{(A.7)}$$

In the latter, we have used a shortened notation for determinants of order $p+1$ (resp. p) under the integral at the r.h.s. of (A.6) (resp. (A.7)).

The convergence of the integrals at the r.h.s. of Eqs (A.6),(A.7) is ensured by the bounds (A.1),(A.3); in fact, by introducing the functions

$$\underline{F}(t,\star;\rho,w,\rho',w',\bullet,\bullet') = \frac{F(t,\star;\rho,w,\rho',w',\bullet,\bullet')}{C_{(\star,\bullet,\bullet')}(1+|t|)^{\hat{N}}(1+\rho)^N(1+|w|)^N(1+\rho')^N(1+|w'|)^N},$$

which are bounded in modulus by 1, one can rewrite the determinants under the integrals of (A.6),(A.7) respectively as follows:

$$C^{p+1}_{(\star,\bullet,\bullet')}(1+|t|)^{\hat{N}(p+1)}(1+\rho)^N(1+|w|)^N(1+\rho')^N(1+|w'|)^N \times \cdots$$

$$\left(\prod_{1\le i\le p}(1+\rho_i)^{2N}(1+|w_i|)^{2N}\right) \left| \begin{matrix} \underline{F}_{\cdot\cdot'} \underline{F}_{\cdot j} \\ \underline{F}_{i\cdot'} \underline{F}_{ij} \end{matrix} \right|_{1\le i,j\le p}, \qquad \text{(A.8)}$$

$$C^{p}_{(\star,\bullet,\bullet')}(1+|t|)^{\hat{N}p} \left(\prod_{1\le i\le p}(1+\rho_i)^{2N}(1+|w_i|)^{2N}\right) |\underline{F}_{ij}|_{1\le i,j\le p}. \qquad \text{(A.9)}$$

In the latter the determinants are similar to those of Eqs (A.6),(A.7), with F replaced by $\underline{F}$; therefore, in view of Hadamard's majorization, they are respectively bounded in modulus by $(p+1)^{\frac{p+1}{2}}$ and $p^{\frac{p}{2}}$. It then follows from (A.8),(A.9) together with the bound (A.3) on G that the integrals in (A.6),(A.7) are absolutely convergent and define functions $N^{(p)}$ and $D^{(p)}$ satisfying the following bounds for all $p \ge 0$:

$$|N^{(p)}(t,\star;\rho,w,\rho',w',\bullet,\bullet')| \le C_{(\star,\bullet,\bullet')}(1+|t|)^{\hat{N}}(1+\rho)^N(1+|w|)^N(1+\rho')^N \times \cdots$$

$$\cdots(1+|w'|)^N(p+1)^{\frac{p+1}{2}} M^p_{(\star,\Gamma)} \left[\frac{(1+|t|)^{\hat{N}}}{|t|^{r-\frac{1}{2}}}\right]^p, \qquad \text{(A.10)}$$

$$|D^{(p)}(t,\star)| \ \leq \ p^{\frac{p}{2}} M^p_{(\star,\Gamma)} \left[\frac{(1+|t|)^{\hat{N}}}{|t|^{r-\frac{1}{2}}} \right]^p . \qquad \text{(A.11)}$$

In the latter, $M_{(\star,\Gamma)}$ denotes a constant (independent of t), which is expressed as follows:

$$M_{(\star,\Gamma)} = c_r^2 C_{(\star,\Gamma)} \times (\mathrm{Area}\Gamma) \int_0^\infty (1+\rho)^{2(N-r)} \rho^{d-1} d\rho \int_{-\infty}^{\infty} (1+|w|)^{2(N-r)} dw, \ \text{(A12)}$$

where we have put $C_{(\star,\Gamma)} = \sup_{(\bullet,\bullet')\in\Gamma\times\Gamma} C_{(\star,\bullet,\bullet')}$.

It now follows from the bounds (A.10),(A.11), that the entire series (A.4),(A.5) are majorized in the whole complex $\alpha-$plane by convergent series; $\mathcal{N}_F$ and $\mathcal{D}_F$ are thus defined as entire functions of α satisfying the following bounds:

$$|\mathcal{N}_F(t,\star;\rho,w,\rho',w',\bullet,\bullet';\alpha)| \leq C_{(\star,\bullet,\bullet')} (1+|t|)^{\hat{N}} (1+\rho)^N (1+|w|)^N (1+\rho')^N \times \cdots$$

$$\cdots (1+|w'|)^N \Phi' \left(|\alpha| M_{(\star,\Gamma)} \frac{(1+|t|)^{\hat{N}}}{|t|^{r-\frac{1}{2}}} \right), \qquad \text{(A.13)}$$

$$|\mathcal{D}_F(t,\star;\alpha) \ - \ 1| \leq \Phi \left(|\alpha| M_{(\star,\Gamma)} \frac{(1+|t|)^{\hat{N}}}{|t|^{r-\frac{1}{2}}} \right), \qquad \text{(A.14)}$$

where

$$\Phi(z) = \sum_{p=1}^{\infty} \frac{z^p}{p!} \ p^{\frac{p}{2}}. \qquad \text{(A.15)}$$

The solution of the Fredholm resolvent equation (A.2) is then obtained as a meromorphic function of α (in $\mathbb{C}$), namely :

$$R_F(t,\star;\rho,w,\rho',w',\bullet,\bullet';\alpha) = \frac{\mathcal{N}_F(t,\star;\rho,w,\rho',w',\bullet,\bullet';\alpha)}{\mathcal{D}_F(t,\star;\alpha)}. \qquad \text{(A.16)}$$

In the applications of this result to the solutions of BS-type equations (of the form $H = B + B \circ H$), the Fredholm parameter α is fixed at either value -1 or $+1$, according to whether H or B is considered as given, so that one has:

$$B = \frac{\mathcal{N}_{H|\alpha=-1}}{\mathcal{D}_{H|\alpha=-1}} = R_{H|\alpha=-1}, \quad H = \frac{\mathcal{N}_{B|\alpha=1}}{\mathcal{D}_{B|\alpha=1}} = R_{B|\alpha=1}. \quad \text{(A.17)}$$

In all cases, it is therefore important to justify the fact that the functions $\mathcal{D}_{H|\alpha=-1}(t,\star)$ and $\mathcal{D}_{B|\alpha=1}(t,\star)$ are not identically equal to zero, which is done below by considering large values of $|t|$: in this connection, the crucial assumption for solving BS-type equations is the choice of a "regularized double-propagator" $G[t;\rho,w]$ satisfying the bounds (A.3).

a) BS-equation in the mass variables and complex angular variables:

We apply the previous analysis to the case when $F \equiv F(t; \rho, w, \rho', w', z, z')$, with $t < 0$ and z, z' varying on a $(d-1)-$cycle Γ of $X_{d-1}^{(c)}$. F is assumed to satisfy a bound of the general form (A.1), with a constant $C_{(\star,\bullet,\bullet')} \equiv C_{(z,z')}$, appropriately chosen (according to (3.27)) as follows:

$$C_{(z,z')} = \mathrm{e}^{N|\Im m\Theta_t|} \times |\sin \Re e\Theta_t|^{-N} \qquad \text{(A.18)}$$

(with $\cos\Theta_t = -z \cdot z'$).

The support of Γ is equal to the Euclidean sphere S_{d-1} in its "initial situation" Γ_0 (see Sec 3.1); it is then distorted in $X_{d-1}^{(c)}$ (e.g. in the way described in the study of $*^{(c)}-$convolution of perikernels in [11]), being always submitted to the condition that $\Gamma \times \Gamma$ belongs to $X_{d-1}^{(c)} \times X_{d-1}^{(c)} \setminus \left(\underline{\Sigma}_s(\zeta, \zeta', t) \cup \underline{\Sigma}_u(\zeta, \zeta', t)\right)$ for all (ζ, ζ', t) (or (t, ρ, w, ρ', w')). This condition takes into account the fact that F is (for each (t, ρ, w, ρ', w')) holomorphic in the domain of a perikernel on $X_{d-1}^{(c)}$ and expresses the requirement that in all the integrals (A.6) the integration space in $(z_1, \ldots, z_p)$ should belong (for each (t, ρ, w, ρ', w')) to the holomorphy domain of the integrand. The use of such integrals (A.6), with integration cycles "floating in complex space" implies the fact (see [13], Proposition 1) that $\mathcal{N}_F(t; \rho, w, \rho', w', z, z'; \alpha)$ is for each fixed values of $(t, \rho, w, \rho', w', \alpha)$ a holomorphic function of (z, z') in the domain covered by the distortion of $\Gamma \times \Gamma$: according to [11], this is the full domain $\mathcal{D}^{(\mathrm{per})} \equiv \{(z, z') \in X_{d-1}^{(c)} \times X_{d-1}^{(c)}; |z \cdot z'| \notin [+1, +\infty[\}$ of a general perikernel.

We now notice that since the integrals $D^{(p)}(t)$ (see Eq (A.7)) do not contain the external variables z, z', it is not worthwhile to distort the integration cycle Γ in the latter. Γ can be kept equal to Γ_0, so that the expression (A.12) yields a fixed numerical constant $M_{(\Gamma_0)} \equiv M_0$, relevant for the bounds (A.11) on the functions $D^{(p)}$ ($(\mathrm{Area}\Gamma_0)$ being equal to the area of the sphere S_{d-1}). It then follows from the bound (A.14) that one has:

$$|\mathcal{D}_{F|\alpha=\pm1}(t) \ - \ 1| \leq \Phi\left(M_0 \frac{(1+|t|)^{\hat{N}}}{|t|^{r-\frac{1}{2}}}\right); \qquad \text{(A.19)}$$

since Φ is an entire function which vanishes at the origin (see Eq (A.15)), and since $\hat{N} - r + \frac{1}{2} < 0$, there exists a value $t = t_1 < 0$ such that *for all* $t \in]-\infty, t_1]$ the function $\mathcal{D}_{F|\alpha=\pm1}(t)$ does *not* vanish (Note that t_1 can be made arbitrarily close to zero if the constant c_r in (A.3) can be taken arbitrarily small). This shows:

Proposition A1

The solution $R_{F|\alpha=\pm1}(t; \rho, w, \rho', w', z, z') = \frac{\mathcal{N}_{F|\alpha=\pm1}(t;\rho,w,\rho',w',z,z')}{\mathcal{D}_{F|\alpha=\pm1}(t)}$ *of the BS-equation with given kernel* $F(t; \rho, w, \rho', w', z, z')$ *and weight* $G[t; \rho, w]$ *satisfying the bounds (A.1),(A.3), is well-defined for* $t \leq t_1, \rho, \rho' \geq 0$, w, w' *real,* *as a holomorphic function of z, z' in the domain $\mathcal{D}^{(\mathrm{per})}$. Under the assumption (A.18) on F, it satisfies a bound of the following form:*

$$|D_{F|\alpha=\pm1}(t)\times R_{F|\alpha=\pm1}(t;\rho,w,\rho',w',z,z')| \leq C_{(z,z')}(1+|t|)^{\hat{N}}(1+\rho)^N(1+|w|)^N\times\cdots$$

$$\cdots(1+\rho')^N(1+|w'|)^N\Phi'\left(M_{(\Gamma)}\frac{(1+|t|)^{\hat{N}}}{|t|^{r-\frac{1}{2}}}\right). \qquad \text{(A.20)}$$

If one now makes use of the assumption that $F(t;\rho,w,\rho',w',z,z') \equiv F([k])$ satisfies the axiomatic analyticity properties of a four-point function [16], one obtains as a by-product of [8,9] the following results:

i) There exists a complex neighborhood $\mathcal{V}$ of the half-line $\{t; t<0\}$ in which $\mathcal{D}_{F|\alpha=\pm1}(t)$ admits an analytic continuation, with $\mathcal{D}_{F|\alpha=\pm1}(t) \neq 0$ for $\Re e t < t_1$ (for some $t_1 < 0$).

ii) There exists a complex neighborhood of the following set: $\{(t,\rho,w,\rho',w',z,z'); t\in\mathcal{V}, \rho\geq0, w\in\mathbb{R}, \rho'\geq0, w'\in\mathbb{R}, (z,z')\in S_{d-1}\times S_{d-1}\}$, in which $\mathcal{N}_{F|\alpha=\pm1}(t;\rho,w,\rho',w',z,z')$ admits an analytic continuation.

These results are obtained (according to [8,9]) by performing a small distortion of the Euclidean integration contour E_{d+1} (here reprsented by the set $\{(\rho,w,z); \rho\geq0,\ w\in\mathbb{R},\ z\in S_{d-1}\}$) inside the axiomatic analyticity domain of F and by using the corresponding properties of analytic continuation of the functions $\mathcal{N}_F$ and $\mathcal{D}_F$.

By now putting together the latter results and those of Proposition A1 and by applying a standard technique of analytic completion (see e.g. Appendix A of [1]), one obtains:

Proposition A2 *The function $\mathcal{N}_{F|\alpha=\pm1}$ (resp. $R_{F|\alpha=\pm1}$) admits an analytic (resp. meromorphic) continuation in a complex neighborhood of the set $\{(t,\rho,w,\rho',w',z,z'); t\in\mathcal{V},\ \rho\geq0, w\in\mathbb{R},\ \rho'\geq0, w'\in\mathbb{R},\ (z,z')\in\mathcal{D}^{(\mathrm{per})}\}$.*

b) BS-equation with dependence on the complex angular momentum variable:

As a second application, we consider the case when $F \equiv F(t,\lambda_t;\rho,w,\rho',w')$, F being holomorphic with respect to λ_t in $\mathbb{C}_+^{(N)}$ for all $t<0$, $\rho\geq0$, $\rho'\geq0$, w,w' real.

By exploiting the bounds (A13) and (A14) similarly as in a) and by taking into account the dependence $C_{(\star,\bullet,\bullet')} = C(\lambda_t)$ of the constant $C_{(\star,\bullet,\bullet')}$ introduced in (A1), we obtain:

Proposition A3 *Let F satisfy the following uniform bound:*

$$|F(t,\lambda_t;\rho,w,\rho',w')| \leq C(\lambda_t)(1+|t|)^{\hat{N}}(1+\rho)^N(1+|w|)^N(1+\rho')^N(1+|w'|)^N \qquad \text{(A 21)}$$

for $t<\hat{t}$ (with $\hat{t}<0$).

Then the solution $R_{F|\alpha=\pm1}(t,\lambda_t;\rho,w,\rho',w') = \frac{\mathcal{N}_{F|\alpha=\pm1}(t,\lambda_t;\rho,w,\rho',w')}{\mathcal{D}_{F|\alpha=\pm1}(t,\lambda_t)}$ *of the BS-equation with given kernel* $F(t,\lambda_t;\rho,w,\rho',w')$ *and with weight* $G[t;\rho,w]$ *satisfying the bound (A3), is well-defined for all* t*, with* $t<\hat{t}$*, as a meromorphic function of* λ_t *in* $\mathbb{C}_+^{(N)}$*, satisfying a uniform majorization of the following form:*

$$|\mathcal{D}_{F|\alpha=\pm1}(t,\lambda_t)\times R_{F|\alpha=\pm1}(t,\lambda_t;\rho,w,\rho',w')| \leq \cdots$$

$$C(\lambda_t)\Phi'\left(\underline{M}_0 C(\lambda_t)\frac{(1+|t|)^{\hat{N}}}{|t|^{r-\frac{1}{2}}}\right)(1+|t|)^{\hat{N}}[(1+\rho)(1+|w|)(1+\rho')(1+|w'|)]^N \tag{A 22}$$

In the general case when $C(\lambda_t)$ *is a locally bounded function , there exists a value* $t=t_C<0$ *depending on* $C(\lambda_t)$ *such that for each* $t<t_C$*,* $R_{F|\alpha=\pm1}$ *is holomorphic in a region of the form* $\{\lambda_t;\ \Re e\lambda_t>N,\ |\lambda_t-N|<\lambda_C(t)\}$*, with* $\lambda_C(t)$ *increasing with* $|t|$ *and tending to infinity for* $t\to-\infty$*.*

Moreover, the following specification holds: Let $C(\lambda_t)=\Psi(|\Im m\lambda_t|)$*, where* Ψ *denotes a bounded positive function on* $[0,\infty[$*, tending to zero at infinity. Then there exists a value* $t=t_\Psi\leq\hat{t}$ *such that for each* $t<t_\Psi$*,* $R_{F|\alpha=\pm1}$ *is holomorphic in* $\mathbb{C}_+^{(N)}$*. Moreover, for* $t_\Psi\leq t<\hat{t}$*,* $R_{F|\alpha=\pm1}$ *is holomorphic in a region of the form* $\{\lambda_t;\ \Re e\lambda_t>N,\ |\Im m\lambda_t|>\nu_\Psi(t)\}$*, with* $\nu_\Psi(t)$ *decreasing with* $|t|$ *and such that* $\nu_\Psi(t_\Psi)=0$*.*

Remark The treatment of the BS-equation (3.26) for partial waves is contained in the latter statement for $\lambda_t=\ell>N$ with $\tilde{f}_\ell(t;\rho,w,\rho',w')=F(t,\ell;\rho,w,\rho',w')$).

Treatment near $t=0$: In a range of the form $\hat{t}\leq t\leq 0$, it is preferable to introduce the variables $W=|t|^{\frac{1}{2}}w$, $W'=|t|^{\frac{1}{2}}w'$, instead of w,w' in Eq. (A.2), and to substitute to (A.1) and (A.3) the following bounds on the corresponding functions $\hat{F}$ and $\hat{G}$, which are consequences (now for $\hat{t}\leq t\leq 0$) of the relevant bounds (3.9), (3.27), (3.28) on H, (4.1)...(4.3) on B and (3.12) on G:

$$|\hat{F}(t,\star;\rho,W,\rho',W',\bullet,\bullet')|\leq C_{(\star,\bullet,\bullet')}(1+|t|)^{\frac{N}{2}}[(1+\rho)(1+|W|)(1+\rho')(1+|W'|)]^N \tag{A.23}$$

$$|\hat{G}[t;\rho,W]|\leq c_r^2 4^r(1+\rho)^{-2r}(1+|W|)^{-r}, \tag{A.24}$$

By repeating all the previous computations with these new variables , one would obtain that for $r>\max(N+\frac{d}{2},2N+1)$, the Fredholm formulae are still applicable in the closed interval $\hat{t}\leq t\leq 0$, the r.h.s. of (A.14) being now replaced by $\Phi\left(|\alpha|M_{(\star,\Gamma)}(1+|t|)^{\frac{N}{2}}\right)$. In particular, the analysis of the possible localization of the poles in the λ_t-plane given in Proposition A3 can be extended to all values of t, with $t\leq 0$.

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